Alternating Direction Optimization for Imaging Inverse Problems

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Outline

Inference/Learning via Optimization

Many inference criteria (in signal processing, machine learning) have the form

$$\widehat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) + \tau c(\mathbf{x})$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ data fidelity, observation model, negative log-likelihood, loss,…

… usually **smooth** and **convex**. Canonical example:

Canonical example: $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|^2$

$c : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ regularization/penalty function, negative log-prior, …

… typically **convex,** often **non-differentiable** (e.g., for sparsity)

Examples: signal/image restoration/reconstruction, sparse representations, compressive sensing/imaging, linear regression, logistic regression, channel sensing, support vector machines, ...

Unconstrained Versus Constrained Optimization

Unconstrained optimization formulation

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} f(\mathbf{x}) + \tau c(\mathbf{x})$$

(Tikhonov regularization)

Constrained optimization formulations

$$\begin{aligned} \hat{\mathbf{x}} \quad &\in \quad \arg\min_{\mathbf{x}} c(\mathbf{x}) \\ &\quad\quad \text{s. t. } \; f(\mathbf{x}) \leq \varepsilon \end{aligned}$$

(Morozov regularization)

$$\begin{aligned} \hat{\mathbf{x}} \quad &\in \quad \arg\min_{\mathbf{x}} f(\mathbf{x}) \\ &\quad\quad \text{s. t. } \; c(\mathbf{x}) \leq \delta \end{aligned}$$

(Ivanov regularization)

"*Equivalent*", under mild conditions; maybe not equally convenient/easy
[Lorenz, 2012]

A Fundamental Dichotomy: Analysis vs Synthesis

[Elad, Milanfar, Rubinstein, 2007], [Selesnick, F, 2010],

$$\widehat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) + \tau c(\mathbf{x})$$

Synthesis regularization:

$\mathbf{x}$ contains **representation** coefficients (not the signal/image itself)

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \mathcal{L}(\mathbf{A}\mathbf{x}) + \tau c(\mathbf{x})$$

$\mathbf{A} = \mathbf{B}\mathbf{W}$, where $\mathbf{B}$ is the observation operator

$\mathbf{W}$ is a synthesis operator; *e.g.*, a *Parseval frame* $\mathbf{W}\mathbf{W}^* = \mathbf{I}$

$\mathcal{L}$ depends on the noise model; *e.g.*, $\mathcal{L}(\mathbf{z}) = \frac{1}{2}\|\mathbf{z} - \mathbf{y}\|_2^2$

typical (sparsity-inducing) regularizer: $c(\mathbf{x}) = \|\mathbf{x}\|_1$

proper, lower semi-continuous (lsc), convex (not strictly), coercive.

A Fundamental Dichotomy: Analysis vs Synthesis (II)

[Elad, Milanfar, Rubinstein, 2007], [Selesnick, F, 2010],

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \mathcal{L}(\mathbf{A}\mathbf{x}) + \tau c(\mathbf{x})$$

Analysis regularization

$\mathbf{x}$ is the signal/image itself, $\mathbf{A}$ is the observation operator

typical frame-based analysis regularizer:

$$c(\mathbf{x}) = \|\mathbf{P}\,\mathbf{x}\|_1$$

analysis operator (e.g., of a Parseval frame, $\mathbf{P}^*\mathbf{P} = \mathbf{I}$)

proper, lsc, convex (not strictly), and coercive.

Total variation (TV) is also "analysis"; proper, lsc, convex (not strictly),
... but not coercive.

Typical Convex Data Terms

Let: $f(\mathbf{x}) = \mathcal{L}(\mathbf{A}\mathbf{x})$ where $\mathcal{L}(\mathbf{z}) \equiv \sum_{i=1}^{m} \xi(z_i, y_i)$

where ξ is one (e.g.) of these functions (log-likelihoods):

Gaussian observations: $\xi_{\mathrm{G}}(z,y) = \frac{1}{2}(z-y)^2 \longrightarrow \mathcal{L}_{\mathrm{G}}$

Poissonian observations: $\xi_{\mathrm{P}}(z,y) = z + \iota_{\mathbb{R}_+}(z) - y\log(z_+) \rightarrow \mathcal{L}_{\mathrm{P}}$

Multiplicative noise: $\xi_{\mathrm{M}}(z,y) = L(z + e^{y-z}) \longrightarrow \mathcal{L}_{\mathrm{M}}$

…all proper, lower semi-continuous (lsc), coercive, convex.

$\mathcal{L}_{\mathrm{G}}$ and $\mathcal{L}_{\mathrm{M}}$ are strictly convex. $\mathcal{L}_{\mathrm{P}}$ is strictly convex if $y_i > 0, \ \forall_i$

A Key Tool: The Moreau Proximity Operator

The *Moreau proximity operator* [Moreau 62], [Combettes, Pesquet, Wajs, 01, 03, 05, 07, 10, 11].

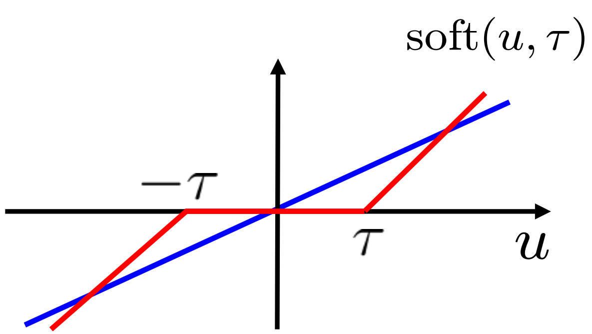
$$\text{prox}_{\tau c}(\mathbf{u}) = \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{x}-\mathbf{u}\|_2^2 + \tau c(\mathbf{x})$$

Classical cases:

Euclidean projection on convex set $\mathcal{C}$

$$c(\mathbf{z}) = \iota_{\mathcal{C}}(\mathbf{z}) = \begin{cases} 0 & \Leftarrow \quad \mathbf{z} \in \mathcal{C} \\ +\infty & \Leftarrow \quad \mathbf{z} \notin \mathcal{C} \end{cases} \Rightarrow \text{prox}_{\tau c}(\mathbf{u}) = \Pi_{\mathcal{C}}(\mathbf{u})$$

$$c(\mathbf{z}) = \tfrac{1}{2}\|\mathbf{z}\|_2^2 \Rightarrow \text{prox}_{\tau c}(\mathbf{u}) = \frac{\mathbf{u}}{1+\tau}$$



$$c(\mathbf{z}) = \|\mathbf{z}\|_1 \Rightarrow \text{prox}_{\tau c}(\mathbf{u}) = \text{soft}(\mathbf{u},\tau) = \text{sign}(\mathbf{u}) \odot \max(|\mathbf{u}| - \tau, 0)$$

Separability: $$c(\mathbf{z}) = \sum_i c_i(z_i) \Rightarrow \left(\text{prox}_{\tau c}(\mathbf{u})\right)_i = \text{prox}_{\tau c_i}(u_i)$$

Moreau Proximity Operators

...many more!

[Combettes, Pesquet, 2010]

	$\phi(x)$	$\mathrm{prox}_\phi x$
i	$\iota_{[\underline{\omega},\overline{\omega}]}(x)$	$P_{[\underline{\omega},\overline{\omega}]}\, x$
ii	$\sigma_{[\underline{\omega},\overline{\omega}]}(x) = \begin{cases} \underline{\omega}x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \overline{\omega}x & \text{otherwise} \end{cases}$	$\mathrm{soft}_{[\underline{\omega},\overline{\omega}]}(x) = \begin{cases} x - \underline{\omega} & \text{if } x < \underline{\omega} \\ 0 & \text{if } x \in [\underline{\omega},\overline{\omega}] \\ x - \overline{\omega} & \text{if } x > \overline{\omega} \end{cases}$
iii	$\psi(x) + \sigma_{[\underline{\omega},\overline{\omega}]}(x)$ $\psi \in \Gamma_0(\mathbb{R})$ differentiable at 0 $\psi'(0) = 0$	$\mathrm{prox}_\psi\big(\mathrm{soft}_{[\underline{\omega},\overline{\omega}]}(x)\big)$
iv	$\max\{\lvert x\rvert - \omega, 0\}$	$\begin{cases} x & \text{if } \lvert x\rvert < \omega \\ \mathrm{sign}(x)\omega & \text{if } \omega \le \lvert x\rvert \le 2\omega \\ \mathrm{sign}(x)(\lvert x\rvert - \omega) & \text{if } \lvert x\rvert > 2\omega \end{cases}$
v	$\kappa\lvert x\rvert^q$	$\mathrm{sign}(x)p$, where $p \ge 0$ and $p + q\kappa p^{q-1} = \lvert x\rvert$
vi	$\begin{cases} \kappa x^2 & \text{if } \lvert x\rvert \le \omega/\sqrt{2\kappa} \\ \omega\sqrt{2\kappa}\lvert x\rvert - \omega^2/2 & \text{otherwise} \end{cases}$	$\begin{cases} x/(2\kappa+1) & \text{if } \lvert x\rvert \le \omega(2\kappa+1)/\sqrt{2\kappa} \\ x - \omega\sqrt{2\kappa}\,\mathrm{sign}(x) & \text{otherwise} \end{cases}$
vii	$\omega\lvert x\rvert + \tau\lvert x\rvert^2 + \kappa\lvert x\rvert^q$	$\mathrm{sign}(x)\mathrm{prox}_{\kappa\lvert\cdot\rvert^q/(2\tau+1)} \dfrac{\max\{\lvert x\rvert - \omega, 0\}}{2\tau+1}$
viii	$\omega\lvert x\rvert - \ln(1 + \omega\lvert x\rvert)$	$(2\omega)^{-1}\,\mathrm{sign}(x)\Big(\omega\lvert x\rvert - \omega^2 - 1 + \sqrt{\big\lvert \omega\lvert x\rvert - \omega^2 - 1\big\rvert^2 + 4\omega\lvert x\rvert}\Big)$
ix	$\begin{cases} \omega x & \text{if } x \ge 0 \\ +\infty & \text{otherwise} \end{cases}$	$\begin{cases} x - \omega & \text{if } x \ge \omega \\ 0 & \text{otherwise} \end{cases}$
x	$\begin{cases} -\omega x^{1/q} & \text{if } x \ge 0 \\ +\infty & \text{otherwise} \end{cases}$	$p^{1/q}$, where $p > 0$ and $p^{2q-1} - xp^{q-1} = q^{-1}\omega$
xi	$\begin{cases} \omega x^{-q} & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}$	$p > 0$ such that $p^{q+2} - xp^{q+1} = \omega q$
xii	$\begin{cases} x\ln(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ +\infty & \text{otherwise} \end{cases}$	$W(e^{x-1})$, where W is the Lambert W-function
xiii	$\begin{cases} -\ln(x - \underline{\omega}) + \ln(-\underline{\omega}) & \text{if } x \in]\underline{\omega}, 0] \\ -\ln(\overline{\omega} - x) + \ln(\overline{\omega}) & \text{if } x \in]0, \overline{\omega}[\\ +\infty & \text{otherwise} \end{cases}$ $\underline{\omega} < 0 < \overline{\omega}$	$\begin{cases} \frac{1}{2}\Big(x + \underline{\omega} + \sqrt{\lvert x - \underline{\omega}\rvert^2 + 4}\Big) & \text{if } x < 1/\underline{\omega} \\ \frac{1}{2}\Big(x + \overline{\omega} - \sqrt{\lvert x - \overline{\omega}\rvert^2 + 4}\Big) & \text{if } x > 1/\overline{\omega} \\ 0 & \text{otherwise} \end{cases}$ (see Figure 1)
xiv	$\begin{cases} -\kappa\ln(x) + \tau x^2/2 + \alpha x & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}$	$\dfrac{1}{2(1+\tau)}\Big(x - \alpha + \sqrt{\lvert x - \alpha\rvert^2 + 4\kappa(1+\tau)}\Big)$
xv	$\begin{cases} -\kappa\ln(x) + \alpha x + \omega x^{-1} & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}$	$p > 0$ such that $p^3 + (\alpha - x)p^2 - \kappa p = \omega$
xvi	$\begin{cases} -\kappa\ln(x) + \omega x^q & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}$	$p > 0$ such that $q\omega p^q + p^2 - xp = \kappa$
xvii	$\begin{cases} -\underline{\kappa}\ln(x - \underline{\omega}) - \overline{\kappa}\ln(\overline{\omega} - x) & \text{if } x \in]\underline{\omega}, \overline{\omega}[\\ +\infty & \text{otherwise} \end{cases}$	$p \in]\underline{\omega}, \overline{\omega}[$ such that $p^3 - (\underline{\omega} + \overline{\omega} + x)p^2 +$ $(\underline{\omega}\overline{\omega} - \underline{\kappa} - \overline{\kappa} + (\underline{\omega} + \overline{\omega})x)p = \underline{\omega}\overline{\omega}x - \underline{\omega}\overline{\kappa} - \overline{\omega}\underline{\kappa}$

Iterative Shrinkage/Thresholding (IST)

$$\widehat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) + \tau c(\mathbf{x})$$

$$\mathbf{x}_{k+1} = \text{prox}_{\tau c/\alpha}\left(\mathbf{x}_k - \frac{1}{\alpha}\nabla f(\mathbf{x}_k)\right)$$

Iterative shrinkage thresholding (IST)
a.k.a. **forward-backward splitting**
a.k.a **proximal gradient algorithm**
[Bruck, 1977], [Passty, 1979], [Lions, Mercier, 1979], [F, Nowak, 01, 03], [Daubechies, Defrise, De Mol, 02, 04], [Combettes and Wajs, 03, 05], [Starck, Candés, Nguyen, Murtagh, 03], [Combettes, Pesquet, Wajs, 03, 05, 07, 11],

Key condition in convergence proofs: ∇f is Lipschtz

…not true, e.g., with Poisson or multiplicative noise.

Not directly applicable with analysis formulations (but see [Loris, Verhoeven, 11])

IST is usually **slow** (specially if τ is small); several accelerated versions:

- Two-step IST (TwIST) [Bioucas-Dias, F, 07]
- Fast IST (FISTA) [Beck, Teboulle, 09], [Tseng, 08]
- Continuation [Hale, Yin, Zhang, 07], [Wright, Nowak, F, 07, 09]
- SpaRSA [Wright, Nowak, F, 08, 09]

Variable Splitting + Augmented Lagrangian

Unconstrained (convex) optimization problem: $$\min_{\mathbf{z}\in\mathbb{R}^d} f_1(\mathbf{z}) + f_2(\mathbf{G}\,\mathbf{z})$$

$c \times d$

Equivalent constrained problem: $$\min_{\mathbf{z}\in\mathbb{R}^d, \mathbf{u}\in\mathbb{R}^c} f_1(\mathbf{z}) + f_2(\mathbf{u})$$
$$\text{s.t.} \quad \mathbf{u} - \mathbf{G}\,\mathbf{z} = 0$$

Augmented Lagrangian (AL):

$$L_\mu(\mathbf{z}, \mathbf{u}, \lambda) = f_1(\mathbf{z}) + f_2(\mathbf{u}) + \lambda^T(\mathbf{G}\mathbf{z} - \mathbf{u}) + \boxed{\frac{\mu}{2}\|\mathbf{G}\mathbf{z} - \mathbf{u}\|_2^2}$$

AL, or method of multipliers [Hestenes, Powell, 1969]

$$(\mathbf{z}_{k+1}, \mathbf{u}_{k+1}) = \arg\min_{\mathbf{z},\mathbf{u}} L_\mu(\mathbf{z}, \mathbf{u}, \lambda_k)$$
$$\lambda_{k+1} = \lambda_k + \mu(\mathbf{G}\,\mathbf{z}_{k+1} - \mathbf{u}_{k+1})$$

equivalent

$$(\mathbf{z}_{k+1}, \mathbf{u}_{k+1}) = \arg\min_{\mathbf{z},\mathbf{u}} f_1(\mathbf{z}) + f_2(\mathbf{u}) + \frac{\mu}{2}\|\mathbf{G}\,\mathbf{z} - \mathbf{u} - \mathbf{d}_k\|_2^2$$
$$\mathbf{d}_{k+1} = \mathbf{d}_k - (\mathbf{G}\,\mathbf{z}_{k+1} - \mathbf{u}_{k+1})$$

Alternating Direction Method of Multipliers (ADMM)

Problem: $\min_{\mathbf{z}\in\mathbb{R}^d} f_1(\mathbf{z}) + f_2(\mathbf{G}\,\mathbf{z})$

Method of multipliers (MM)

$$(\mathbf{z}_{k+1}, \mathbf{u}_{k+1}) = \arg\min_{\mathbf{z},\mathbf{u}} f_1(\mathbf{z}) + f_2(\mathbf{u}) + \frac{\mu}{2}\|\mathbf{G}\,\mathbf{z} - \mathbf{u} - \mathbf{d}_k\|_2^2$$

$$\mathbf{d}_{k+1} = \mathbf{d}_k - (\mathbf{G}\,\mathbf{z}_{k+1} - \mathbf{u}_{k+1})$$

ADMM [Glowinski, Marrocco, 75], [Gabay, Mercier, 76], [Gabay, 83], [Eckstein, Bertsekas, 92]

$$\mathbf{z}_{k+1} = \arg\min_{\mathbf{z}} f_1(\mathbf{z}) + \frac{\mu}{2}\|\mathbf{G}\,\mathbf{z} - \mathbf{u}_k - \mathbf{d}_k\|^2$$

$$\mathbf{u}_{k+1} = \arg\min_{\mathbf{u}} f_2(\mathbf{u}) + \frac{\mu}{2}\|\mathbf{G}\,\mathbf{z}_{k+1} - \mathbf{u} - \mathbf{d}_k\|^2$$

$$\mathbf{d}_{k+1} = \mathbf{d}_k - (\mathbf{G}\,\mathbf{z}_{k+1} - \mathbf{u}_{k+1})$$

Interpretations: variable splitting + augmented Lagrangian + NLBGS;

Douglas-Rachford splitting on the dual [Eckstein, Bertsekas, 92]

split-Bregman approach [Goldstein, Osher, 08]

A *Cornerstone* Result on ADMM

[Eckstein, Bertsekas, 1992]

The problem $$\min_{\mathbf{z}\in\mathbb{R}^d} \; f_1(\mathbf{z}) + f_2(\mathbf{G}\,\mathbf{z})$$

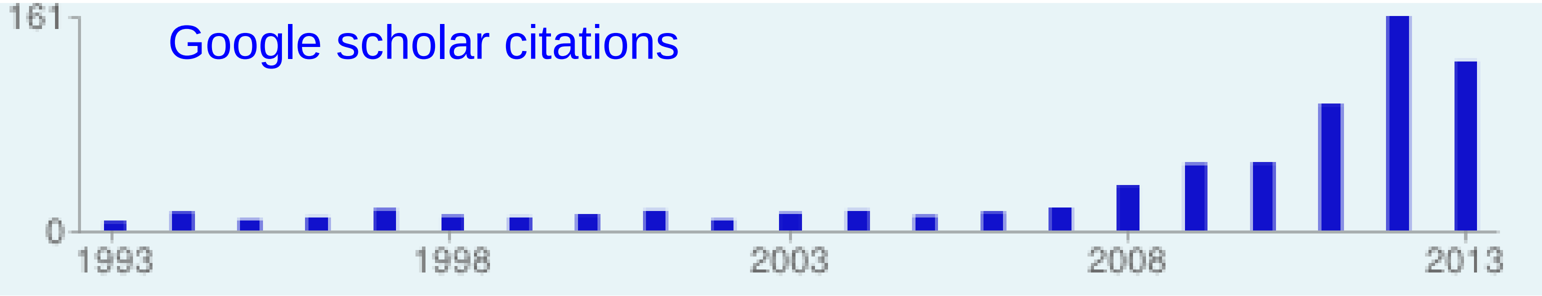
f_1, f_2 closed, proper, convex; $\mathbf{G}$ full column rank.

$(\mathbf{z}_k, \;\; k = 0, 1, 2, ...)$ is the sequence produced by ADMM, with $\mu > 0$

then, if the problem has a solution, say $\bar{\mathbf{z}}$, then

$$\lim_{k\rightarrow\infty} \mathbf{z}_k = \bar{\mathbf{z}}$$

Inexact minimizations allowed, as long as the errors are absolutely summable).



Explosion of applications in signal processing, machine learning, statistics, ...

[Giovanneli, Coulais, 05], Giannakis et al, 08, 09,...], [Tomioka et al, 09], [Boyd et al, 11], [Goldfarb, Ma, 10,...], [Fessler et al, 11, ...], [Mota et al, 10], [Jakovetić et al, 12], [Banerjee et al, 12], [Esser, 09], [Ng et al, 20], [Setzer, Steidl, Teuber, 09], [Yang, Zhang, 11], [Combettes, Pesquet, 10,...], [Chan, Yang, Yuan, 11], ...

(The Art of) Applying ADMM

Synthesis formulation: $$\min_{\mathbf{x}} \mathcal{L}(\mathbf{BWx}) + \tau c(\mathbf{x})$$

Template problem for ADMM $$\min_{\mathbf{z}} \; f_1(\mathbf{z}) + f_2(\mathbf{G}\,\mathbf{z})$$

Naïve mapping: $$\mathbf{G} = \mathbf{BW}, \quad f_1 = \tau c, \quad f_2 = \mathcal{L}$$

ADMM

$$\mathbf{z}_{k+1} = \arg\min_{\mathbf{z}} \; \tau\, c(\mathbf{z}) + \frac{\mu}{2}\|\mathbf{B}\,\mathbf{W}\,\mathbf{z} - \mathbf{u}_k - \mathbf{d}_k\|^2$$

usually hard!

$$\mathbf{u}_{k+1} = \arg\min_{\mathbf{u}} \; \mathcal{L}(\mathbf{u}) + \frac{\mu}{2}\|\mathbf{B}\,\mathbf{W}\,\mathbf{z}_{k+1} - \mathbf{u} - \mathbf{d}_k\|^2$$

usually easy $\text{prox}_{\mathcal{L}/\mu}$

$$\mathbf{d}_{k+1} = \mathbf{d}_k - (\mathbf{B}\,\mathbf{W}\,\mathbf{z}_{k+1} - \mathbf{u}_{k+1})$$

Applying ADMM

Analysis formulation: $\min_{\mathbf{x}} \mathcal{L}(\mathbf{B}\mathbf{x}) + \tau c(\mathbf{P}\mathbf{x})$

Template problem for ADMM $\min_{\mathbf{z}} \; f_1(\mathbf{z}) + f_2(\mathbf{G}\,\mathbf{z})$

Naïve mapping: $\mathbf{G} = \mathbf{P}, \quad f_1 = \mathcal{L} \circ \mathbf{B}, \quad f_2 = \tau\, c$

$$\mathbf{z}_{k+1} = \arg\min_{\mathbf{z}} \; \mathcal{L}(\mathbf{B}\,\mathbf{z}) + \frac{\mu}{2}\|\mathbf{P}\,\mathbf{z} - \mathbf{u}_k - \mathbf{d}_k\|^2$$

$$\mathbf{u}_{k+1} = \arg\min_{\mathbf{u}} \; \tau c(\mathbf{u}) + \frac{\mu}{2}\|\mathbf{P}\,\mathbf{z}_{k+1} - \mathbf{u} - \mathbf{d}_k\|^2$$

$$\mathbf{d}_{k+1} = \mathbf{d}_k - (\mathbf{P}\,\mathbf{z}_{k+1} - \mathbf{u}_{k+1})$$

usually easy $\mathrm{prox}_{\tau\, c/\mu}$

Easy if: $\mathcal{L}$ is quadratic and $\mathbf{B}$ and $\mathbf{P}$ diagonalized by common transform (e.g., DFT) (split-Bregman [Goldstein, Osher, 08])

Applying ADMM

Analysis formulation: $\min_{\mathbf{x}} \mathcal{L}(\mathbf{B}\mathbf{x}) + \tau c(\mathbf{P}\mathbf{x})$

Template problem for ADMM $\min_{\mathbf{z}} \; f_1(\mathbf{z}) + f_2(\mathbf{G}\,\mathbf{z})$

Naïve mapping: $\mathbf{G} = \mathbf{B}, \quad f_1 = \tau\, c \circ \mathbf{P}, \quad f_2 = \mathcal{L}$

$$\mathbf{z}_{k+1} = \arg\min_{\mathbf{z}} \tau\, c(\mathbf{P}\,\mathbf{z}) + \frac{\mu}{2}\|\mathbf{B}\,\mathbf{z} - \mathbf{u}_k - \mathbf{d}_k\|^2$$

$$\mathbf{u}_{k+1} = \arg\min_{\mathbf{u}} \; \mathcal{L}(\mathbf{u}) + \frac{\mu}{2}\|\mathbf{B}\,\mathbf{z}_{k+1} - \mathbf{u} - \mathbf{d}_k\|^2$$

usually easy $\text{prox}_{\mathcal{L}/\mu}$

$$\mathbf{d}_{k+1} = \mathbf{d}_k - (\mathbf{B}\,\mathbf{z}_{k+1} - \mathbf{u}_{k+1})$$

Easy if: c is quadratic and
$\mathbf{B}$ and $\mathbf{P}$ diagonalized by common transform (e.g., DFT)

General Template for ADMM with Two or More Functions

[F and Bioucas-Dias, 2009]

Consider a more general problem

$$\min_{\mathbf{z}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{z}) \qquad (P)$$

$g_j : \mathbb{R}^{p_j} \rightarrow \bar{\mathbb{R}}$

Proper, closed, convex functions

$\mathbf{H}^{(j)} \in \mathbb{R}^{p_j \times d}$

Arbitrary matrices

There are many ways to write (P) as

$$\min_{\mathbf{z}\in\mathbb{R}^d} f_1(\mathbf{z}) + f_2(\mathbf{G}\,\mathbf{z})$$

We propose:

$$f_1(\mathbf{z}) = 0, \qquad \mathbf{G} = \begin{bmatrix} \mathbf{H}^{(1)} \\ \vdots \\ \mathbf{H}^{(J)} \end{bmatrix}, \qquad f_2\left(\begin{bmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(J)} \end{bmatrix}\right) = \sum_{j=1}^{J} g_j(\mathbf{u}^{(j)})$$

ADMM for Two or More Functions

$$\min_{\mathbf{z}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{z}), \qquad \min_{\mathbf{z}\in\mathbb{R}^d} \; f_2(\mathbf{G}\,\mathbf{z}), \qquad \mathbf{G} = \begin{bmatrix} \mathbf{H}^{(1)} \\ \vdots \\ \mathbf{H}^{(J)} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(J)} \end{bmatrix}$$

$$\mathbf{z}\mathbf{z}_{k+1} = \left(\sum_{j=1}^{J} (\mathbf{H}^{(j)})^* \mathbf{H}^{(j)} \right)^{-1} \sum_{j=1}^{J} (\mathbf{H}^{(j)})^* \left(\mathbf{u}_k^{(j)} + \mathbf{d}_k^{(j)} \right)$$

$$\mathbf{u}_{k+1}^{(1)} = \arg\min_{\mathbf{u}} g_1(\mathbf{u}) + \frac{\mu}{2} \|\mathbf{u} - \mathbf{H}^{(1)}\mathbf{z}_{k+1} + \mathbf{d}_k^{(1)}\|^2 = \mathrm{prox}_{g_1/\mu}(\mathbf{H}^{(1)}\mathbf{z}_{k+1} - \mathbf{d}_k^{(j)})$$

$$\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots$$

$$\mathbf{u}_{k+1}^{(J)} = \arg\min_{\mathbf{u}} g_J(\mathbf{u}) + \frac{\mu}{2} \|\mathbf{u} - \mathbf{H}^{(J)}\mathbf{z}_{k+1} + \mathbf{d}_k^{(J)}\|^2 = \mathrm{prox}_{g_J/\mu}(\mathbf{H}^{(J)}\mathbf{z}_{k+1} - \mathbf{d}_k^{(J)})$$

$$\mathbf{d}_{k+1}^{(1)} = \mathbf{d}_k^{(1)} - (\mathbf{H}^{(1)}\mathbf{z}_{k+1} - \mathbf{u}_{k+1}^{(1)})$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\mathbf{d}_{k+1}^{(J)} = \mathbf{d}_k^{(J)} - (\mathbf{H}^{(J)}\mathbf{z}_{k+1} - \mathbf{u}_{k+1}^{(J)})$$

ADMM for Two or More Functions

$$\mathbf{z}_{k+1} = \left(\sum_{j=1}^{J} (\mathbf{H}^{(j)})^* \, \mathbf{H}^{(j)} \right)^{-1} \sum_{j=1}^{J} (\mathbf{H}^{(j)})^* \left(\mathbf{u}_k^{(j)} + \mathbf{d}_k^{(j)} \right)$$

$$\mathbf{u}_{k+1}^{(1)} = \text{prox}_{g_1/\mu}(\mathbf{H}^{(1)} \mathbf{z}_{k+1} - \mathbf{d}_k^{(j)})$$

$$\vdots$$

$$\mathbf{u}_{k+1}^{(J)} = \text{prox}_{g_1/\mu}(\mathbf{H}^{(J)} \mathbf{z}_{k+1} - \mathbf{d}_k^{(j)})$$

$$\mathbf{d}_{k+1}^{(1)} = \mathbf{d}_k^{(1)} - (\mathbf{H}^{(1)} \mathbf{z}_{k+1} - \mathbf{u}_{k+1}^{(1)})$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\mathbf{d}_{k+1}^{(J)} = \mathbf{d}_k^{(J)} - (\mathbf{H}^{(J)} \mathbf{z}_{k+1} - \mathbf{u}_{k+1}^{(J)})$$

Conditions for easy applicability:

inexpensive proximity operators

inexpensive matrix inversion

...a cursing and a blessing!

ADMM for Two or More Functions

Applies to sum of convex terms

$$\min_{\mathbf{z}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{z})$$

Computation of proximity operators is parallelizable

Handling of matrices is isolated in a pure quadratic problem

Conditions for easy applicability: inexpensive proximity operators

inexpensive matrix inversion

Matrix inversion may be a *curse or a blessing!* (more later)

Similar algorithm: *simultaneous directions method of multipliers* (SDMM)

[Setzer, Steidl, Teuber, 2010], [Combettes, Pesquet, 2010]

Other ADMM versions for more than two functions

[Hong, Luo, 2012, 2013], [Ma, 2012]

Linear/Gaussian Observations: Frame-Based Analysis

Problem: $$\widehat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2 + \tau\,\|\mathbf{P}\mathbf{x}\|_1$$

Template: $$\min_{\mathbf{z}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{z})$$

Mapping: $$J=2,\quad g_1(\mathbf{z}) = \frac{1}{2}\|\mathbf{z}-\mathbf{y}\|_2^2, \qquad g_2(\mathbf{z}) = \tau\,\|\mathbf{z}\|_1$$

$$\mathbf{H}^{(1)} = \mathbf{A}, \qquad \mathbf{H}^{(2)} = \mathbf{P},$$

Convergence conditions: g_1 and g_2 are closed, proper, and convex.

$$\mathbf{G} = \begin{bmatrix} \mathbf{A} \\ \mathbf{P} \end{bmatrix}$$ has full column rank.

Resulting algorithm: SALSA

(*split augmented Lagrangian shrinkage algorithm*) [Afonso, Bioucas-Dias, F, 2009, 2010]

ADMM for the Linear/Gaussian Problem: SALSA

Key steps of SALSA (both for analysis and synthesis):

Moreau proximity operator of $g_1(\mathbf{z}) = \frac{1}{2}\|\mathbf{z}-\mathbf{y}\|_2^2,$

$$\text{prox}_{g_1/\mu}(\mathbf{u}) = \arg\min_{\mathbf{z}} \frac{1}{2\mu}\|\mathbf{z}-\mathbf{y}\|_2^2 + \frac{1}{2}\|\mathbf{z}-\mathbf{u}\|_2^2 = \frac{\mathbf{y}+\mu\,\mathbf{u}}{1+\mu}$$

Moreau proximity operator of $g_2(\mathbf{z}) = \tau\|\mathbf{z}\|_1,$

$$\text{prox}_{g_2/\mu}(\mathbf{u}) = \text{soft}\Big(\mathbf{u}, \tau/\mu\Big)$$

Matrix inversion:

$$\mathbf{z}_{k+1} = \Big[\mathbf{A}^*\mathbf{A} + \mathbf{P}^*\mathbf{P}\Big]^{-1}\Bigg(\mathbf{A}^*\Big(\mathbf{u}_k^{(1)} + \mathbf{d}_k^{(1)}\Big) + \mathbf{P}^*\Big(\mathbf{u}_k^{(2)} + \mathbf{d}_k^{(2)}\Big)\Bigg)$$

...next slide!

Handling the Matrix Inversion: Frame-Based Analysis

Frame-based analysis: $\left[\mathbf{A}^*\mathbf{A} + \mathbf{P}^*\mathbf{P}\right]^{-1} = \left[\mathbf{A}^*\mathbf{A} + \mathbf{I}\right]^{-1}$

$\mathbf{P}^*\mathbf{P} = \mathbf{I}$

Parseval frame

diagonal → ← DFT (FFT)

Periodic deconvolution: $\mathbf{A} = \mathbf{U}^*\mathbf{D}\mathbf{U}$

$O(n \log n)$

$$\left[\mathbf{A}^*\mathbf{A} + \mathbf{I}\right]^{-1} = \mathbf{U}^*\left[|\mathbf{D}|^2 + \mathbf{I}\right]^{-1}\mathbf{U}$$

subsampling matrix: $\mathbf{M}\mathbf{M}^* = \mathbf{I}$

Compressive imaging (MRI): $\mathbf{A} = \mathbf{M}\mathbf{U}$

$O(n \log n)$

$$\left[\mathbf{U}^*\mathbf{M}^*\mathbf{M}\mathbf{U} + \mathbf{I}\right]^{-1} = \mathbf{I} - \frac{1}{2}\mathbf{U}^*\mathbf{M}^*\mathbf{M}\mathbf{U}$$

matrix inversion lemma

subsampling matrix: $\mathbf{S}^*\mathbf{S}$ is diagonal

Inpainting (recovery of lost pixels): $\mathbf{A} = \mathbf{S}$

$O(n)$

$\left[\mathbf{S}^*\mathbf{S} + \mathbf{I}\right]^{-1}$ is a diagonal inversion

SALSA for Frame-Based Synthesis

Problem: $\widehat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau \|\mathbf{x}\|_1$

$\mathbf{A} = \mathbf{B}\mathbf{W}$ — observation matrix ($\mathbf{B}$), synthesis matrix ($\mathbf{W}$)

Template: $\min_{\mathbf{z}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{z})$

Mapping: $J = 2$, $\quad g_1(\mathbf{z}) = \frac{1}{2}\|\mathbf{z} - \mathbf{y}\|_2^2, \quad g_2(\mathbf{z}) = \tau \|\mathbf{z}\|_1$

$$\mathbf{H}^{(1)} = \mathbf{A} = \mathbf{B}\mathbf{W} \qquad \mathbf{H}^{(2)} = \mathbf{I},$$

Convergence conditions: g_1 and g_2 are closed, proper, and convex.

$$\mathbf{G} = \begin{bmatrix} \mathbf{B}\,\mathbf{W} \\ \mathbf{I} \end{bmatrix}$$ has full column rank.

Handling the Matrix Inversion: Frame-Based Synthesis

Frame-based analysis: $\left[\sum_{j=1}^{J} (\mathbf{H}^{(j)})^* \mathbf{H}^{(j)}\right]^{-1} = \left[\mathbf{W}^*\mathbf{B}^*\mathbf{B}\mathbf{W} + \mathbf{I}\right]^{-1}$

DFT

Periodic deconvolution: $\mathbf{B} = \mathbf{U}^*\mathbf{D}\mathbf{U}$

diagonal matrix

$O(n \log n)$

$$\left[\mathbf{W}^*\mathbf{B}^*\mathbf{B}\mathbf{W} + \mathbf{I}\right]^{-1} = \mathbf{I} - \mathbf{W}^*\mathbf{U}^*\mathbf{D}^*\left[|\mathbf{D}|^2 + \mathbf{I}\right]^{-1}\mathbf{D}\mathbf{U}\mathbf{W}$$

matrix inversion lemma + $\mathbf{W}\mathbf{W}^* = \mathbf{I}$

subsampling matrix: $\mathbf{M}\mathbf{M}^* = \mathbf{I}$

Compressive imaging (MRI): $\mathbf{B} = \mathbf{M}\mathbf{U}$

$O(n \log n)$

$$\left[\mathbf{W}^*\mathbf{U}^*\mathbf{M}^*\mathbf{M}\mathbf{U}\mathbf{W} + \mathbf{I}\right]^{-1} = \mathbf{I} - \frac{1}{2}\mathbf{W}^*\mathbf{U}^*\mathbf{M}^*\mathbf{M}\mathbf{U}\mathbf{W}$$

subsampling matrix: $\mathbf{S}\mathbf{S}^* = \mathbf{I}$

Inpainting (recovery of lost pixels): $\mathbf{B} = \mathbf{S}$

$O(n \log n)$

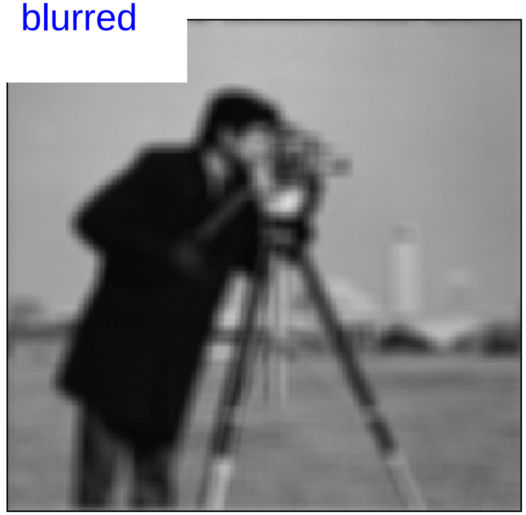
$$\left[\mathbf{W}^*\mathbf{S}^*\mathbf{S}\mathbf{W} + \mathbf{I}\right]^{-1} = \mathbf{I} - \frac{1}{2}\mathbf{W}^*\mathbf{S}^*\mathbf{S}\mathbf{W}^*$$

SALSA Experiments

9x9 uniform blur,

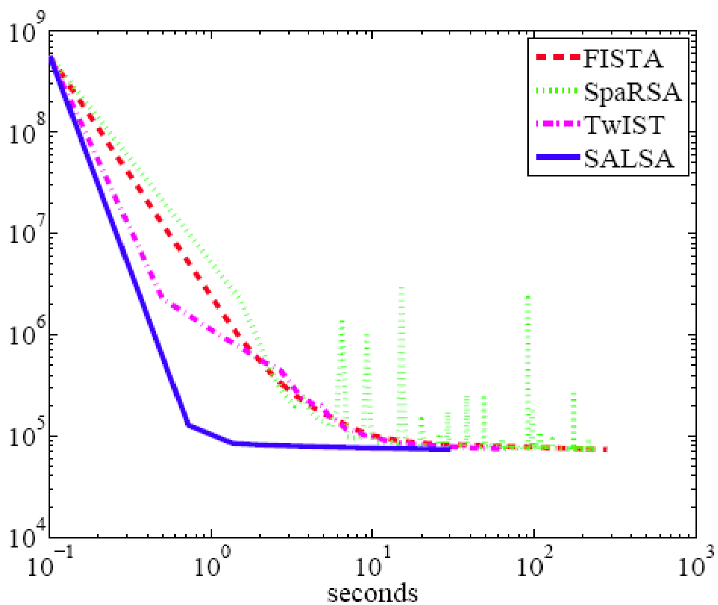
40dB BSNR

blurred

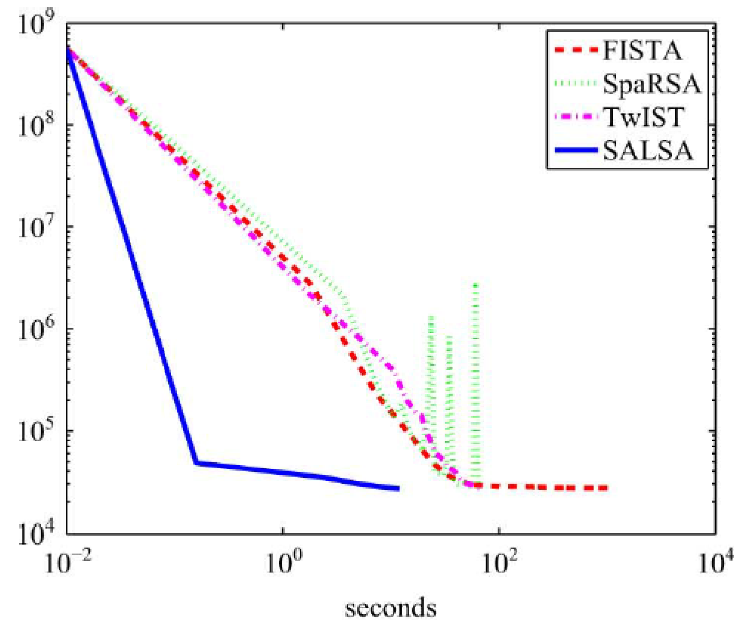


restored

undecimated Haar frame, ℓ_1 regularization.



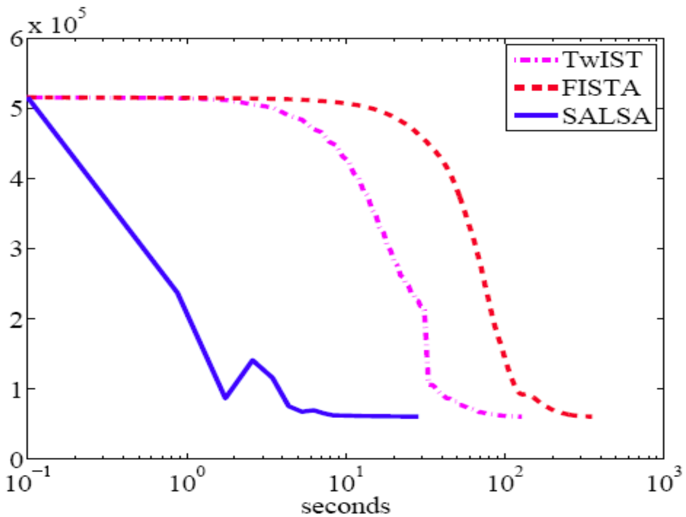
TV regularization



SALSA Experiments

Image inpainting

(50% missing)



Alg.	Calls to $\mathbf{B}, \mathbf{B}^H$	Iter.	CPU time (sec.)	MSE	ISNR (dB)
FISTA	1022	340	263.8	92.01	18.96
TwIST	271	124	112.7	100.92	18.54
SALSA	84	28	20.88	77.61	19.68

Conjecture: SALSA is fast because it's ***blessed*** by the matrix inversion;

e.g., $\mathbf{A}^*\mathbf{A} + \mathbf{I}$ is the (regularized) Hessian of the data term;

...second-order (curvature) information (Newton, Levenberg-Maquardt)

Frame-Based Analysis Deconvolution of Poissonian Images

Problem template: $$\min_{\mathbf{u}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{u}) \qquad (P1)$$

positivity constraint

Frame-analysis regularization: $$\widehat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \mathcal{L}_{\mathrm{P}}(\mathbf{B}\,\mathbf{x}) + \lambda\|\mathbf{P}\,\mathbf{x}\|_1 + \iota_{\mathbb{R}^n_+}(\mathbf{x})$$

Same form as $(P1)$ with: $$J = 3, \quad g_1 = \mathcal{L}_{\mathrm{P}}, \quad g_2 = \|\cdot\|_1, \quad g_3 = \iota_{\mathbb{R}^n_+}$$

Convergence conditions: g_1, g_2, and g_3 are closed, proper, and convex.

$$\mathbf{G} = \begin{bmatrix} \mathbf{B} \\ \mathbf{P} \\ \mathbf{I} \end{bmatrix}$$ has full column rank

Required inversion: $$\left[\mathbf{B}^*\mathbf{B} + \mathbf{P}^*\mathbf{P} + \mathbf{I}\right]^{-1} = \left[\mathbf{B}^*\mathbf{B} + 2\,\mathbf{I}\right]^{-1}$$

…again, easy in periodic deconvolution, MRI, inpainting, …

Proximity Operator of the Poisson Log-Likelihood

Proximity operator of the Poisson log-likelihood

$$\mathrm{prox}_{\mathcal{L}/\mu}(\mathbf{u}) = \arg\min_{\mathbf{z}} \sum_i \xi(z_i, y_i) + \frac{\mu}{2}\|\mathbf{z} - \mathbf{u}\|_2^2$$

$$\xi(z, y) = z + \iota_{\mathbb{R}_+}(z) - y \log(z_+)$$

Separable problem with closed-form (non-negative) solution

[Combettes, Pesquet, 09, 11]:

$$\mathrm{prox}_{\xi(\cdot,y)}(u) = \frac{1}{2}\left(u - \frac{1}{\mu} + \sqrt{\big(u - (1/\mu)\big)^2 + 4\,y/\mu}\,\right)$$

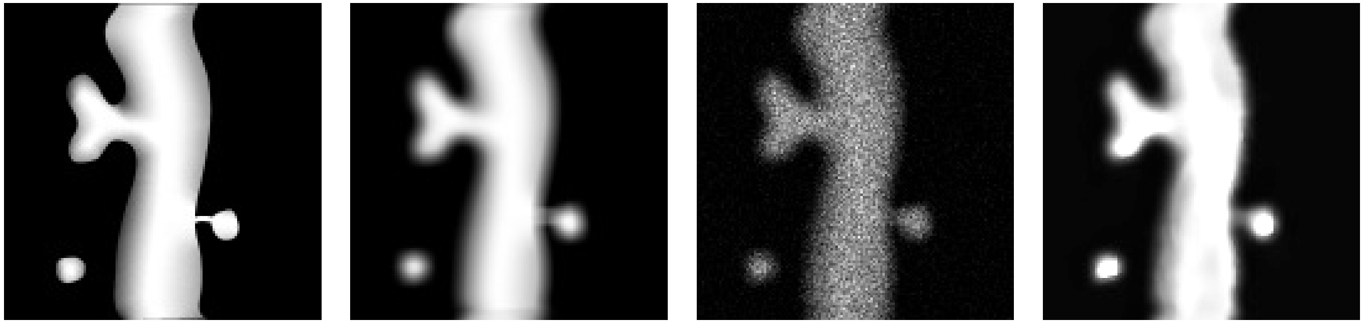
Proximity operator of $g_3 = \iota_{\mathbb{R}^n_+}$ is simply $\mathrm{prox}_{\iota_{\mathbb{R}^n_+}}(\mathbf{x}) = (\mathbf{x})_+$

Experiments

PIDAL = Poisson image deconvolution by augmented Lagrangian
[F and Bioucas-Dias, 2010]

Comparison with [Dupé, Fadili, Starck, 09] and [Starck, Bijaoui, Murtagh, 95]

		PIDAL-TV			PIDAL-FA			[Dupé, Fadili, Starck, 09]			[Starck et al, 95]
Image	M	MAE	iterations	time	MAE	iterations	time	MAE	iterations	time	MAE
Cameraman	5	0.27	120	22	**0.26**	70	13	0.35	6	4.5	0.37
Cameraman	30	1.29	51	9.1	**1.22**	39	7.4	1.47	98	75	2.06
Cameraman	100	3.99	33	6.0	**3.63**	36	6.8	4.31	426	318	5.58
Cameraman	255	8.99	32	5.8	**8.45**	37	7.0	10.26	480	358	12.3
Neuron	5	**0.17**	117	3.6	0.18	66	2.9	0.19	6	3.9	0.19
Neuron	30	**0.68**	54	1.8	0.77	44	2.0	0.82	161	77	0.95
Neuron	100	**1.75**	43	1.4	2.04	41	1.8	2.32	427	199	2.88
Neuron	255	3.52	43	1.4	**3.47**	42	1.9	5.25	202	97	6.31
Cell	5	0.12	56	10	**0.11**	36	7.6	0.12	6	4.5	0.12
Cell	30	0.57	31	6.5	0.54	39	8.2	0.56	85	64	**0.47**
Cell	100	1.71	85	15	1.46	31	6.4	1.72	215	162	**1.37**
Cell	255	3.77	89	17	3.33	34	7.0	5.45	410	308	**3.10**



$$\text{MAE} \equiv \frac{\|\widehat{\mathbf{x}} - \mathbf{x}\|_1}{n}$$

Morozov Formulation

Unconstrained optimization formulation: $$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau c(\mathbf{x})$$

Constrained optimization (Morozov) formulation:

$$\min_{\mathbf{x}} \quad c(\mathbf{x})$$
$$\text{s.t.} \quad \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \leq \varepsilon$$

basis pursuit denoising, if $c(\mathbf{x}) = \|\mathbf{x}\|_1$

[Chen, Donoho, Saunders, 1998]

Both analysis and synthesis can be used:

- frame-based analysis, $c(\mathbf{x}) = \|\mathbf{P}\mathbf{x}\|_1$
- frame-based synthesis $c(\mathbf{x}) = \|\mathbf{x}\|_1$

$$\mathbf{A} = \mathbf{B}\,\mathbf{W}$$

Proposed Approach for Constrained Formulation

Constrained problem: $$\min_{\mathbf{x}} \; c(\mathbf{x})$$
$$\text{s.t.} \; \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \leq \varepsilon$$

…can be written as $$\min_{\mathbf{x}} \; c(\mathbf{x}) + \iota_{\mathcal{B}(\varepsilon,\mathbf{y})}(\mathbf{A}\,\mathbf{x})$$

$$\mathcal{B}(\varepsilon,\mathbf{y}) = \{\mathbf{x} \in \mathbb{R}^n \; : \; \|\mathbf{x} - \mathbf{y}\|_2 \; \square \; \varepsilon\}$$

…which has the form $$\min_{\mathbf{u}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{u}) \qquad (P1)$$

with $J = 2, \quad g_1(\mathbf{z}) = c(\mathbf{z}), \qquad \mathbf{H}^{(1)} = \mathbf{I}$

$g_2(\mathbf{z}) = \iota_{E(\varepsilon,\mathbf{y})}(\mathbf{z}), \qquad \mathbf{H}^{(2)} = \mathbf{A}$

$$\mathbf{G} = \begin{bmatrix} \mathbf{I} \\ \mathbf{A} \end{bmatrix}$$

full column rank

Resulting algorithm: C-SALSA (constrained-SALSA)

[Afonso, Bioucas-Dias, F, 2010,2011]

Some Aspects of C-SALSA

Moreau proximity operator of $\iota_{\mathcal{B}(\varepsilon,\mathbf{y})}$ is simply a projection on an ℓ_2 ball:

$$\mathrm{prox}_{\iota_{\mathcal{B}(\varepsilon,\mathbf{y})}}(\mathbf{u}) = \arg\min_{\mathbf{z}} \iota_{\mathcal{B}(\varepsilon,\mathbf{y})} + \frac{1}{2}\|\mathbf{z}-\mathbf{u}\|_2^2$$

$$= \begin{cases} \mathbf{u} & \Leftarrow \quad \|\mathbf{u}-\mathbf{y}\|_2 \leq \varepsilon \\ \mathbf{y} + \frac{\varepsilon(\mathbf{u}-\mathbf{y})}{\|\mathbf{u}-\mathbf{y}\|_2} & \Leftarrow \quad \|\mathbf{u}-\mathbf{y}\|_2 > \varepsilon \end{cases}$$

As SALSA, also C-SALSA involves a marix inverse

$$\left[\mathbf{W}^*\mathbf{B}^*\mathbf{B}\mathbf{W} + \mathbf{I}\right]^{-1} \quad \text{or} \quad \left[\mathbf{B}^*\mathbf{B} + \mathbf{P}^*\mathbf{P}\right]^{-1}$$

…all the same tricks as above.

C-SALSA Experiments: Image Deblurring

Image deconvolution benchmark problems:

Experiment	blur kernel	σ^2
1	9×9 uniform	0.56^2
2A	Gaussian	2
2B	Gaussian	8
3A	$h_{ij} = 1/(1 + i^2 + j^2)$	2
3B	$h_{ij} = 1/(1 + i^2 + j^2)$	8

NESTA: [Becker, Bobin, Candès, 2011]

SPGL1: [van den Berg, Friedlander, 2009]

Frame-synthesis

Expt.	Avg. calls to $\mathbf{B}, \mathbf{B}^H$ (min/max)			Iterations			CPU time (seconds)		
	SPGL1	NESTA	C-SALSA	SPGL1	NESTA	C-SALSA	SPGL1	NESTA	C-SALSA
1	1029 (659/1290)	3520 (3501/3541)	398 (388/406)	340	880	134	441.16	590.79	100.72
2A	511 (279/663)	4897 (4777/4981)	451 (442/460)	160	1224	136	202.67	798.81	98.85
2B	377 (141/532)	3397 (3345/3473)	362 (355/370)	98	849	109	120.50	557.02	81.69
3A	675 (378/772)	2622 (2589/2661)	172 (166/175)	235	656	58	266.41	423.41	42.56
3B	404 (300/475)	2446 (2401/2485)	134 (130/136)	147	551	41	161.17	354.59	29.57

Frame-analysis

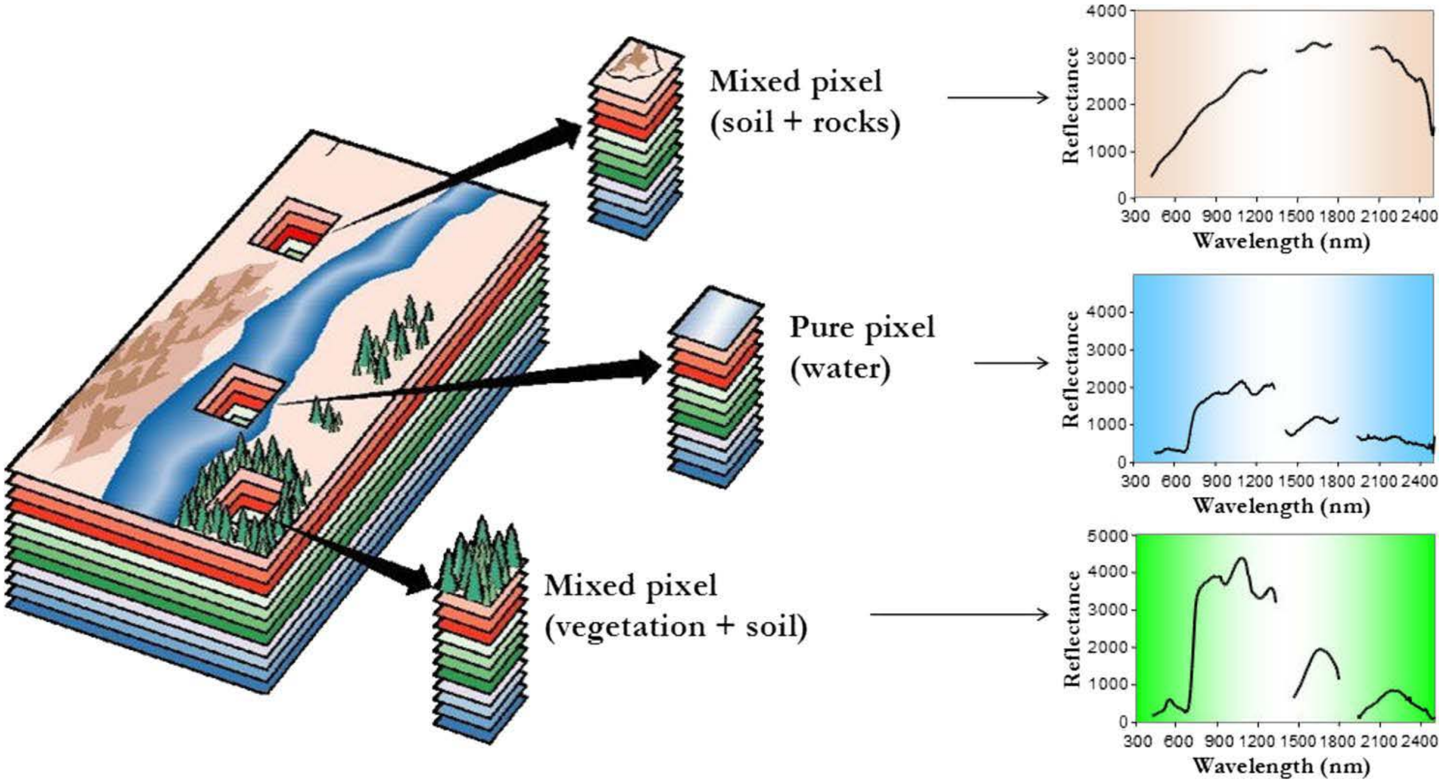
Expt.	Avg. calls to $\mathbf{B}, \mathbf{B}^H$ (min/max)		Iterations		CPU time (seconds)	
	NESTA	C-SALSA	NESTA	C-SALSA	NESTA	C-SALSA
1	2881 (2861/2889)	413 (404/419)	720	138	353.88	80.32
2A	2451 (2377/2505)	362 (344/371)	613	109	291.14	62.65
2B	2139 (2065/2197)	290 (278/299)	535	87	254.94	50.14
3A	2203 (2181/2217)	137 (134/143)	551	42	261.89	23.83
3B	1967 (1949/1985)	116 (113/119)	492	39	236.45	22.38

Total-variation

Expt.	Avg. calls to $\mathbf{B}, \mathbf{B}^H$ (min/max)		Iterations		CPU time (seconds)	
	NESTA	C-SALSA	NESTA	C-SALSA	NESTA	C-SALSA
1	7783 (7767/7795)	695 (680/710)	1945	232	311.98	62.56
2A	7323 (7291/7351)	559 (536/578)	1830	150	279.36	38.63
2B	6828 (6775/6883)	299 (269/329)	1707	100	265.35	25.47
3A	6594 (6513/6661)	176 (98/209)	1649	59	250.37	15.08
3B	5514 (5417/5585)	108 (104/110)	1379	37	210.94	9.23

Spectral Unmixing

[Bioucas-Dias, F, 10]



Goal: find the relative abundance of each "material" in each pixel.

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\,\mathbf{x} - \mathbf{y}\|_2^2 + \iota_{\mathbb{R}_+^n}(\mathbf{x}) + \iota_{\{1\}}(\mathbf{1}^T\mathbf{x})$$

Given library of spectra

indicator of the canonical simplex

Spectral Unmixing

Problem: $\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\,\mathbf{x} - \mathbf{y}\|_2^2 + \iota_{\mathbb{R}_+^n}(\mathbf{x}) + \iota_{\{1\}}(\mathbf{1}^T\mathbf{x})$

Template: $\min_{\mathbf{u}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{u}) \qquad (P1)$

Mapping: $J = 3, \quad g_1(\mathbf{z}) = \frac{1}{2}\|\mathbf{z} - \mathbf{y}\|_2^2,$ $\quad g_2(\mathbf{z}) = \iota_{\mathbb{R}_+^n}(\mathbf{z})$, $\quad g_3(z) = \iota_{\{1\}}(z)$

$$\mathbf{H}^{(1)} = \mathbf{A}, \qquad \mathbf{H}^{(2)} = \mathbf{I}, \qquad \mathbf{H}^{(3)} = \mathbf{1}^T$$

Proximity operators are trivial.

Matrix inversion: $\left[\sum_{j=1}^{J}(\mathbf{H}^{(j)})^*\,\mathbf{H}^{(j)}\right]^{-1} = \left[\mathbf{A}^T\mathbf{A} + \mathbf{I} + \mathbf{1}\mathbf{1}^T\right]^{-1}$

…can be precomputed; typical sizes 200~300 x 500~1000 (bands x library size)

Non-Periodic Deconvolution

Analysis formulation for deconvolution $\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau c(\mathbf{x})$

ADMM / SALSA easy (only?) if $\mathbf{A}$ is circulant (periodic convolution - FFT)

Periodicity is an **artificial** assumption

…as are other boundary conditions (BC)

Neumann

Dirichlet

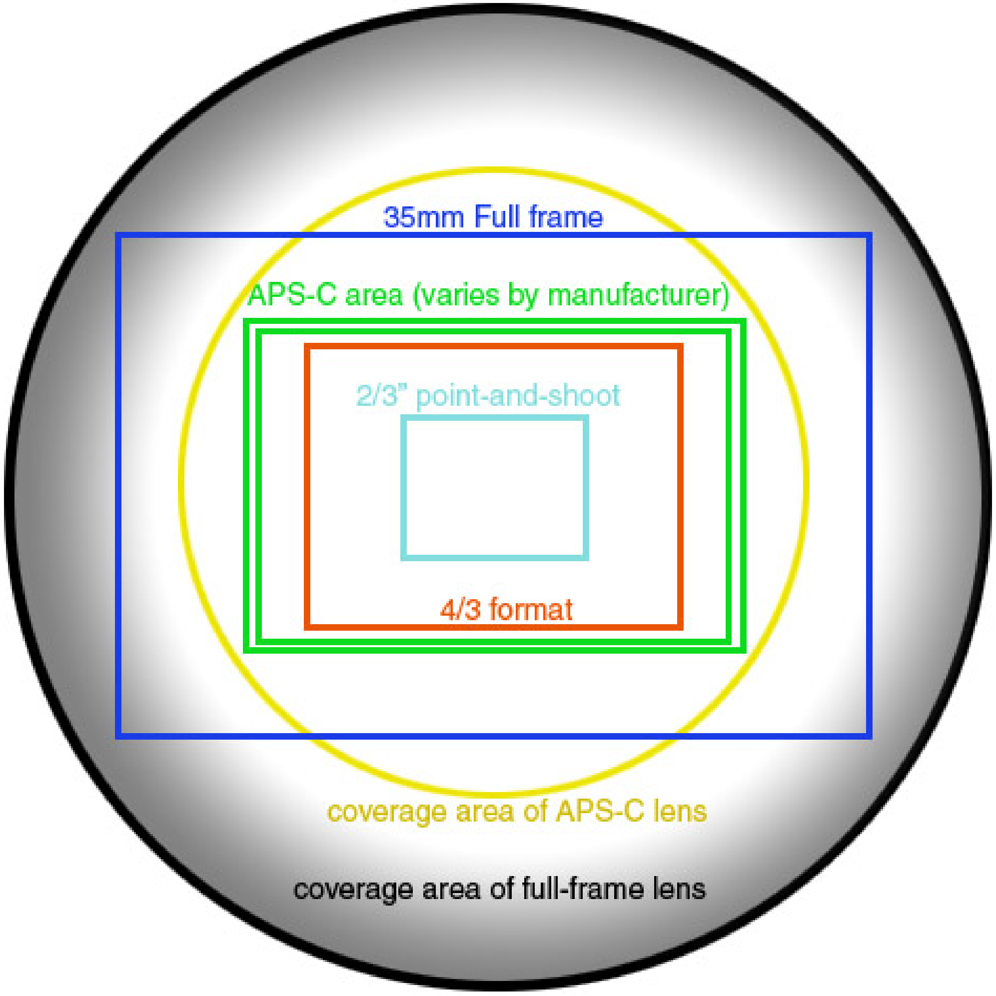
$\mathbf{A}$ is (block) circulant

$\mathbf{A}$ is (block) Toeplitz + Hankel

[Ng, Chan, Tang, 1999]

$\mathbf{A}$ is (block) Toeplitz

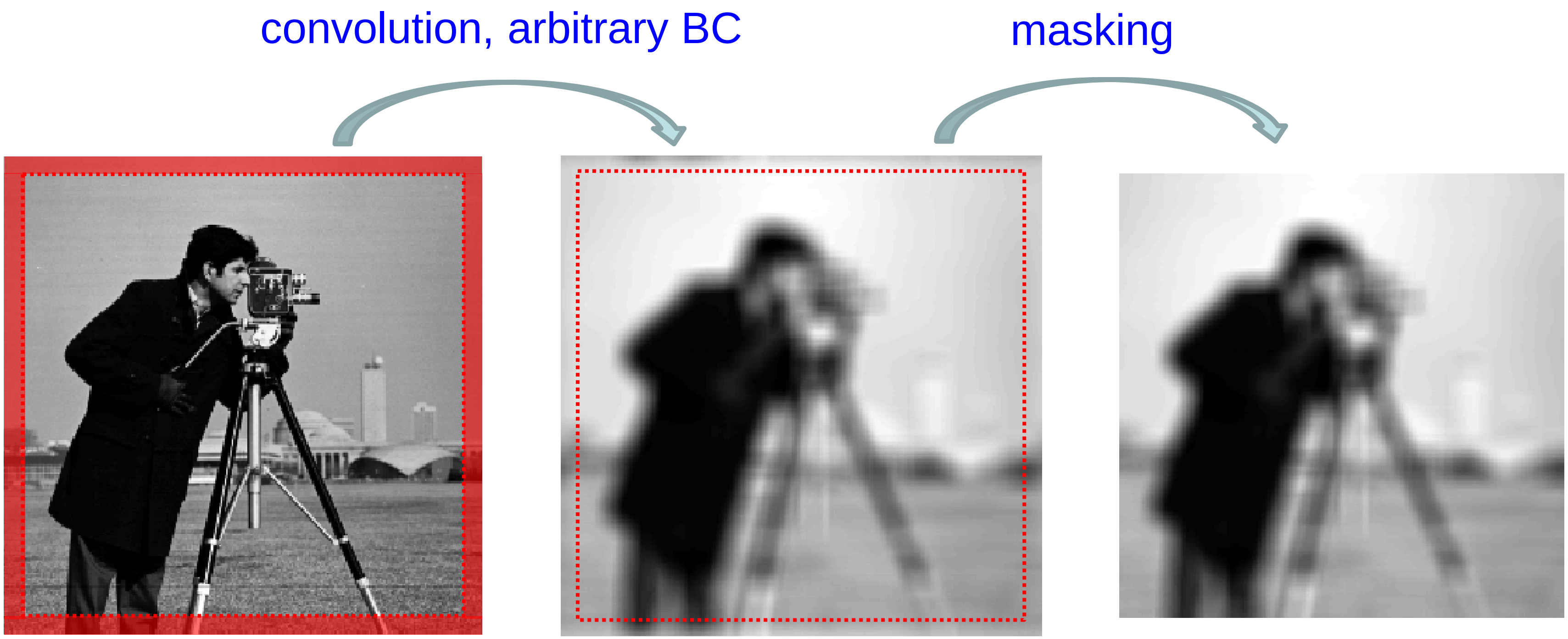
Why Periodic, Neumann, Dirichlet Boundary Conditions are “wrong”



Non-Periodic Deconvolution

The natural choice: the boundary is unknown

[Chan, Yip, Park, 05], [Reeves, 05], [Sorel, 12], [Almeida, F, 12,13], [Matakos, Ramani, Fessler, 12, 13]



unknown values

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{M}\mathbf{B}\mathbf{x} - \mathbf{y}\|_2^2 + \tau c(\mathbf{x})$$

mask — periodic convolution

Non-Periodic Deconvolution (Frame-Analysis)

Problem: $$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{MBx} - \mathbf{y}\|_2^2 + \tau\|\mathbf{Px}\|_1$$

Template: $$\min_{\mathbf{z}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{z})$$

Naïve mapping: $J = 2$, $g_1(\mathbf{z}) = \frac{1}{2}\|\mathbf{z} - \mathbf{y}\|_2^2, \quad g_2(\mathbf{z}) = \tau\,\|\mathbf{z}\|_1$

$$\mathbf{H}^{(1)} = \mathbf{M}\,\mathbf{B} \qquad \mathbf{H}^{(2)} = \mathbf{P},$$

Difficulty: need to compute $\left[\mathbf{B}^*\mathbf{M}^*\mathbf{MB} + \mathbf{P}^*\mathbf{P}\right]^{-1} = \left[\mathbf{B}^*\mathbf{M}^*\mathbf{MB} + \mathbf{I}\right]^{-1}$

...the tricks above are no longer applicable.

Non-Periodic Deconvolution (Frame-Analysis)

Problem: $$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{MBx} - \mathbf{y}\|_2^2 + \tau\|\mathbf{Px}\|_1$$

Template: $$\min_{\mathbf{z}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{z})$$

Better mapping: $$J = 2, \quad g_1(\mathbf{z}) = \frac{1}{2}\|\mathbf{Mz} - \mathbf{y}\|_2^2, \quad g_2(\mathbf{z}) = \tau\,\|\mathbf{z}\|_1$$

$$\mathbf{H}^{(1)} = \mathbf{B} \qquad \mathbf{H}^{(2)} = \mathbf{P},$$

$$\left[\mathbf{B}^*\mathbf{B} + \mathbf{P}^*\mathbf{P}\right]^{-1} = \left[\mathbf{B}^*\mathbf{B} + \mathbf{I}\right]^{-1}$$

easy via FFT ($\mathbf{B}$ is circulant)

$$\mathrm{prox}_{g_1/\mu}(\mathbf{u}) = \arg\min_{\mathbf{z}} \frac{1}{2\mu}\|\mathbf{Mz} - \mathbf{y}\|_2^2 + \frac{1}{2}\|\mathbf{z} - \mathbf{u}\|_2^2$$

$$= \left(\mathbf{M}^T\mathbf{M} + \mu\mathbf{I}\right)^{-1}\left(\mathbf{M}^T\mathbf{y} + \mu\mathbf{u}\right)$$

diagonal

Non-Periodic Deconvolution: Example (19x19 uniform blur)

original (256 × 256)

observed (238 × 238)

Proposed

FA-MD (ISRN = 10.63dB)

Assuming periodic BC

FA-BC (ISNR = -2.52dB)

Edge tapering

FA-ET (ISNR = 3.06dB)

Non-Periodic Deconvolution: Example (19x19 motion blur)

original (256×256)

observed (238×238)

Assuming periodic BC

TV-BC (ISNR = 0.91dB)

Edge tapering

TV-ET (ISNR = 9.38dB)

Proposed

TV-MD (ISNR = 12.59dB)

Non-Periodic Deconvolution + Inpainting

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{M}\mathbf{B}\mathbf{x} - \mathbf{y}\|_2^2 + \tau c(\mathbf{x})$$

Mask the boundary
and the missing pixels

periodic convolution

original (256×256)

observed (238×238)

Also applicable to super-resolution
(ongoing work)

FA-CG (SNR = 20.58dB)

FA-MD (SNR = 20.57dB)

Non-Periodic Deconvolution via Accelerated IST

The syntesis formulation is easily handled by IST (or FISTA, TwIST, SpaRSA,...)
[Matakos, Ramani, Fessler, 12, 13]

periodic convolution

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{MBWx} - \mathbf{y}\|_2^2 + \tau\|\mathbf{x}\|_1$$

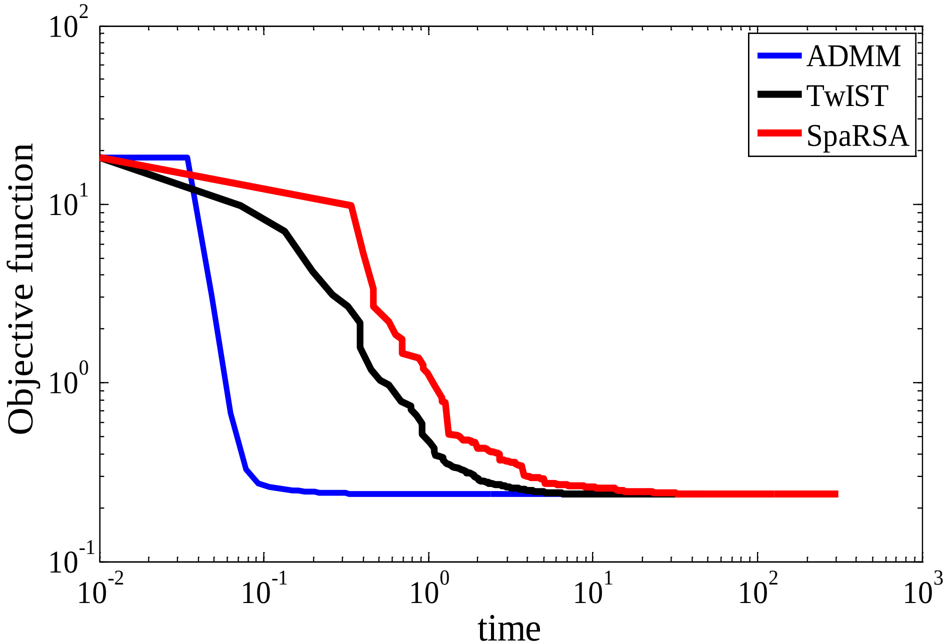
mask

Parseval frame synthesis

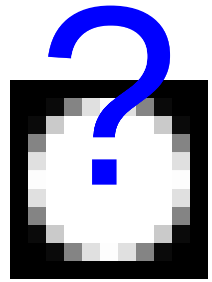
Ingredients: $\text{prox}_{\tau\|\cdot\|_1}(\mathbf{u}) = \text{soft}(\mathbf{u}, \tau)$

$$\nabla \tfrac{1}{2}\|\mathbf{MBWx} - \mathbf{y}\|_2^2 = \mathbf{W}^*\mathbf{B}^*\mathbf{M}^*\left(\mathbf{MBWx} - \mathbf{y}\right)$$

(analysis formulation cannot be addressed by IST, FIST, SpaRSA, TwIST,...)



Into the Non-convex Realm: Blind Image Deconvolution (BID)

 =  $*$  $+$ noise

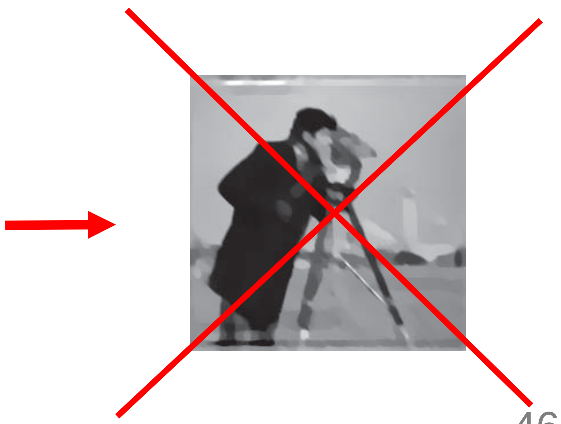
Degradation model: $\mathbf{y} = \mathbf{h} * \mathbf{x} + \mathbf{n}$

Difficulties:

Ill-posed : - infinite number of solutions.

- ill-conditioned blurring operator.

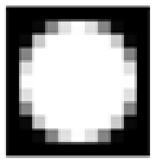
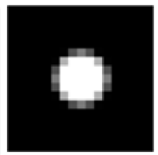
Unknown boundaries (usually ignored)



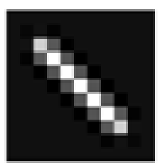
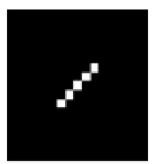
BID Methods and Rescrtictions on the Blurrig Filter

Hard restrictions:
(parameterized filters)

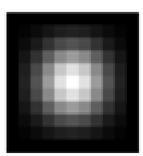
- circular blurs:
  [Yin *et al*, 06]

- linear blurs:
  [Krahmer *et al*, 06]
  [Oliveira *et al*, 07]

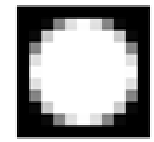
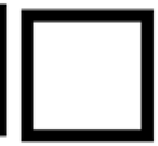



- Gaussian blurs:
  [Rooms *et al*, 04]
  [Krylov *et al*, 09]

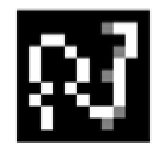
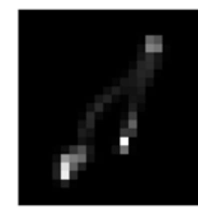


Soft restrictions:
(regularized filters)

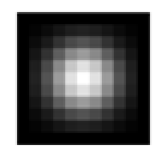
- TV regularization:
  [Babacan *et al* 09],
  [Amizic el all 10], [li,12]

- Sparse regularization:
  [Fregus *et al,* 06];
  [Levin *et al*, 09, 11]
  [Shan el al, 08], [Cho, 09]
  [Krishnan, 11],[Xu, 11], [Cai, 12]

- Smooth regularization:
  [Joshi *et al*, 08;
  Babacan *et al*, 09]



Into the Non-convex Realm: Blind Image Deconvolution (BID)

$$\mathbf{y} = \mathbf{h} * \mathbf{x} + \mathbf{n}$$

Both $\mathbf{x}$ and $\mathbf{h}$ are unknown

Objective function (non-convex):

$$C_\lambda(\mathbf{x}, \mathbf{h}) = \frac{1}{2}\|\mathbf{y} - \mathbf{M}\ \mathbf{B}\ \mathbf{x}\|_2^2 + \lambda \underbrace{\sum_{i=1}^{m} (\|\mathbf{F}_i\ \mathbf{x}\|_2)^q}_{\Phi(\mathbf{x})} + \iota_{\mathcal{S}+}(\mathbf{h})$$

Support and positivity

Boundary mask ($\mathbf{M}$)

Matrix representation of the convolution with $\mathbf{h}$ ($\mathbf{B}$)

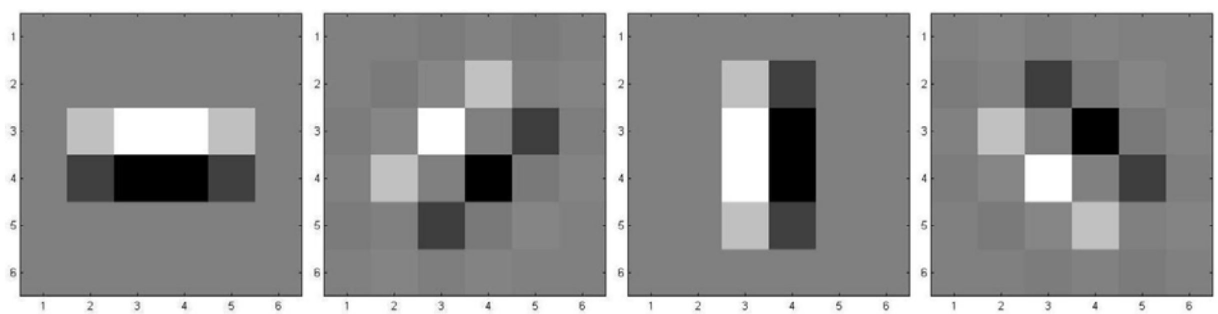
[Almeida and F, 13]

$\Phi(\mathbf{x})$ is "enhanced" TV; $q \in (0,\ 1]$ (typically 0.5);

$\mathbf{F}_i$ is the convolution with four "edge filters" at location i

$\mathbf{F}_i \in \mathbb{R}^{4\times m}$

$\mathbf{F}_i\,\mathbf{x} \in \mathbb{R}^4$



Blind Image Deconvolution (BID)

Algorithm 1: Continuation-based BID.

Set $\widehat{\mathbf{h}}$ to the identity filter, $\widehat{\mathbf{x}} = \mathbf{y}$ and $\lambda = \lambda_0$; choose $\alpha < 1$.
repeat
  $\widehat{\mathbf{x}} \leftarrow \arg\min_{\mathbf{x}} \mathbf{C}_\lambda(\mathbf{x}, \widehat{\mathbf{h}})$ update image estimate
  $\widehat{\mathbf{h}} \leftarrow \arg\min_{\mathbf{h}} \mathbf{C}_\lambda(\widehat{\mathbf{x}}, \mathbf{h})$,   update blur estimate
  $\lambda \leftarrow \alpha\, \lambda$
until *stopping criterion is satisfied*

[Almeida et al, 2010, 2013]

Updating the image estimate

$$\widehat{\mathbf{x}} \leftarrow \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{y} - \mathbf{M}\mathbf{H}\mathbf{x}\|^2 + \lambda\Phi(\mathbf{x})$$

Standard image deconvolution, with unknown boundaries; ADMM as above.

Blind Image Deconvolution (BID)

Updating the image estimate

$$\widehat{\mathbf{x}} \leftarrow \arg\min_{\mathbf{x}\in\mathbb{R}^m} \frac{1}{2}\|\mathbf{y} - \mathbf{M}\mathbf{B}\mathbf{x}\|^2 + \lambda \sum_{i=1}^{m} \left(\|\mathbf{F}_i \mathbf{x}\|_2\right)^q$$

Template: $\min_{\mathbf{z}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{z})$

Mapping: $J = m+1, \quad g_i(\mathbf{z}) = \|\mathbf{z}\|_2^q, \;\; i = 1, ..., m,$

$$\mathbf{H}^{(i)} = \mathbf{F}_i, \;\; i = 1, ..., m,$$

$$g_{m+1}(\mathbf{z}) = \tfrac{1}{2}\|\mathbf{M}\mathbf{z} - \mathbf{y}\|_2^2, \quad \mathbf{H}^{(m+1)} = \mathbf{B}$$

All the matrices are circulant: matrix inversion step in ADMM easy with FFT.

Also possible to compute $\operatorname{prox}_{\tau\,\|\cdot\|_2^q}(\mathbf{u}) = \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{x} - \mathbf{u}\|_2^2 + \tau\,\|\mathbf{x}\|_2^q$

for $q \in \left\{0, \tfrac{1}{2}, \tfrac{2}{3}, 1, \tfrac{4}{3}, \tfrac{3}{2}, 2\right\}$

Algorithm 1: Continuation-based BID.

Set $\widehat{\mathbf{h}}$ to the identity filter, $\widehat{\mathbf{x}} = \mathbf{y}$ and $\lambda = \lambda_0$; choose $\alpha < 1$.
repeat
- $\widehat{\mathbf{x}} \leftarrow \arg\min_{\mathbf{x}} \mathbf{C}_\lambda(\mathbf{x}, \widehat{\mathbf{h}})$ — update image estimate
- $\widehat{\mathbf{h}} \leftarrow \arg\min_{\mathbf{h}} \mathbf{C}_\lambda(\widehat{\mathbf{x}}, \mathbf{h})$, — update blur estimate
- $\lambda \leftarrow \alpha\, \lambda$

until *stopping criterion is satisfied*

Updating the blur estimate: notice that $\mathbf{h} * \mathbf{x} = \mathbf{H}\mathbf{x} = \mathbf{X}\mathbf{h}$

$$\widehat{\mathbf{h}} \leftarrow \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{y} - \mathbf{M}\mathbf{X}\mathbf{h}\|^2 + \iota_{\mathcal{S}^+}(\mathbf{h})$$

Like standard image deconvolution, with a support and positivity constraint.

Prox of support and positivity constraint is trivial: $\text{prox}_{\iota_{\mathcal{S}^+}}(\mathbf{h}) = \Pi_{\mathcal{S}^+}(\mathbf{h})$

Blind Image Deconvolution (BID)

Algorithm 1: Continuation-based BID.

1 Set $\widehat{\mathbf{h}}$ to the identity filter, $\widehat{\mathbf{x}} = \mathbf{y}$ and $\lambda = \lambda_0$; choose $\alpha < 1$.
2 **repeat**
3 $\quad \widehat{\mathbf{x}} \leftarrow \arg\min_{\mathbf{x}} \mathbf{C}_\lambda(\mathbf{x}, \widehat{\mathbf{h}})$
4 $\quad \widehat{\mathbf{h}} \leftarrow \arg\min_{\mathbf{h}} \mathbf{C}_\lambda(\widehat{\mathbf{x}}, \mathbf{h})$,
5 $\quad \lambda \leftarrow \alpha\, \lambda$
6 **until** *stopping criterion is satisfied*

Question: when to stop? What value of λ to choose?

For non-blind deconvolution, many approaches for choosing λ

generalized cross validation, L-curve, SURE and variants thereof

[Bertero, Poggio, Torre, 88], [Thomson, Brown, Kay, Titterington, 92], [Galatsanos, Kastagellos, 92], [Hansen, O'Leary, 93], [Eldar, 09], [Giryes, Elad, Eldar 11], [Luisier, Blu, Unser 09], [Ramani, Blu, Unser, 10], [Ramani, Rosen, Nielsen, Fessler, 12],...

Bayesian methods (some for BID)

[Babacan, Molina, Katsaggelos, 09], [Fergus et al, 06], [Amizic, Babacan, Molina, Katsaggelos, 10], [Chantas, Galatsanos, Molina, Katsaggelos, 10], [Oliveira, Bioucas-Dias, F, 09]

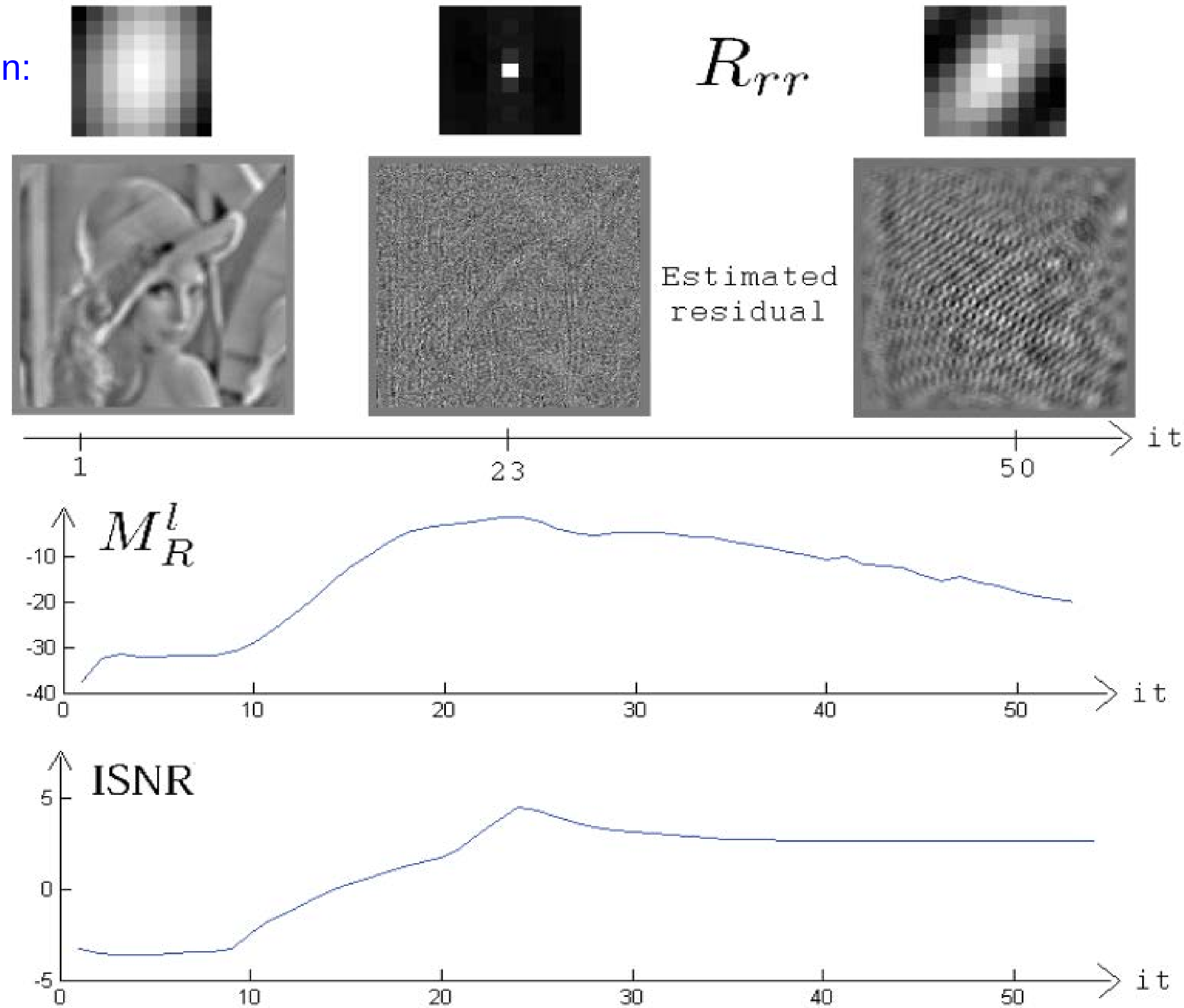
No-reference quality measures [Lee, Lai, Chen, 07], [Zhu, Milanfar, 10]

Blind Image Deconvolution: Stopping Criterion

Proposed rationale: if the blur kernel is well estimated, the residual is white.

Autocorrelation:

Whiteness:



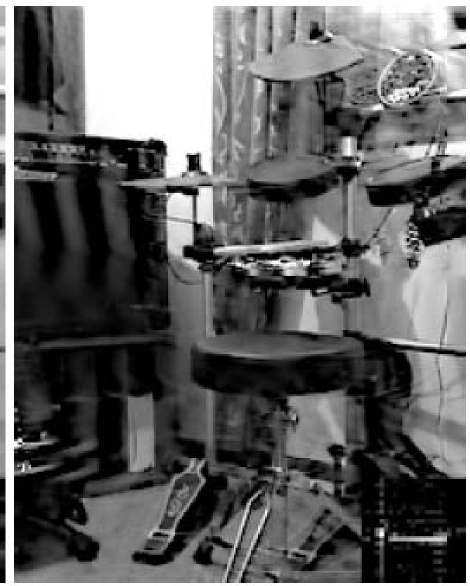
Blind Image Deconvolution (BID)

Experiment with real motion blurred photo

Blurred photo

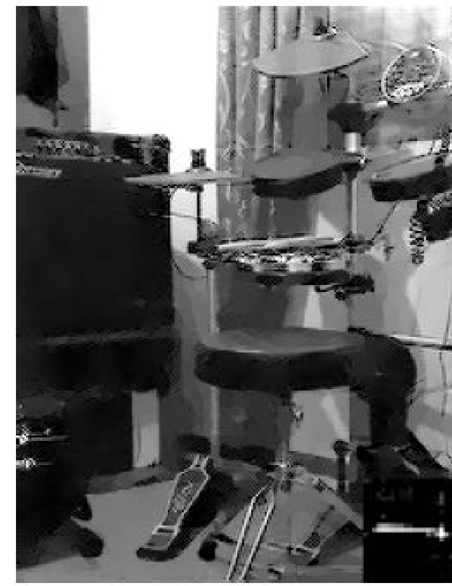
[14], 70 seconds

[Krishnan et al, 2011]



[16], 100 seconds

[Levin et al, 2011]



Proposed method, 55 seconds

Blind Image Deconvolution (BID)

Experiment with real out-of-focus photo

Observed photo.

[Almeida et al, 2010]

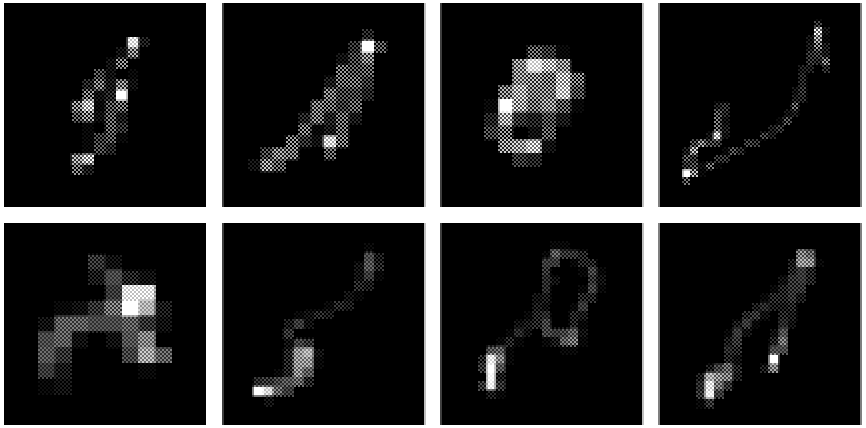
proposed

Blind Image Deconvolution (BID): Synthetic Results

Realistic motion blurs:

[Levin, Weiss, Durant, Freeman, 09]

Images: Lena, Cameraman



Average results over 2 images and 8 blurs:

	Method	∞dB	40dB	30dB	
ISNR* (dB)	[31]	6.14	5.90	4.91	[Krishnan et al, 11]
	[35]	5.51	5.72	4.79	[Levin et al, 11]
	[50]	4.70	4.70	4.30	[Xu, Jia, 10]
	Ours	**9.00**	**8.43**	**6.70**	
Time (s)	[31]	80	66	62	[Krishnan et al, 11]
	[35]	399	399	399	[Levin et al, 11]
	[50]	1.5^2	1.5^2	1.5^2	[Xu, Jia, 10] (GPU)
	Ours	**70**	**55**	**45**	

Blind Image Deconvolution (BID): Handling Staurations

Several digital images have saturated pixels (at 0 or max): this impacts BID!

Easy to handle in our approach: just mask them out

$$\min\left(\alpha\,\mathbf{x} * \mathbf{h}, 255\right)$$

ignoring saturations

knowing saturations

out-of-focus (disk) blur

Summary:

- Alternating direction optimization (ADMM) is powerful, versatile, modular.
- Main hurdle: need to solve a linear system (invert a matrix) at each iteration…
- …however, sometimes this turns out to be an advantage.
- State of the art results in several image/signal reconstruction problems.

Thanks!