Hyperspectral Unmixing Via Sparse Regression Optimization Problems and Algorithms

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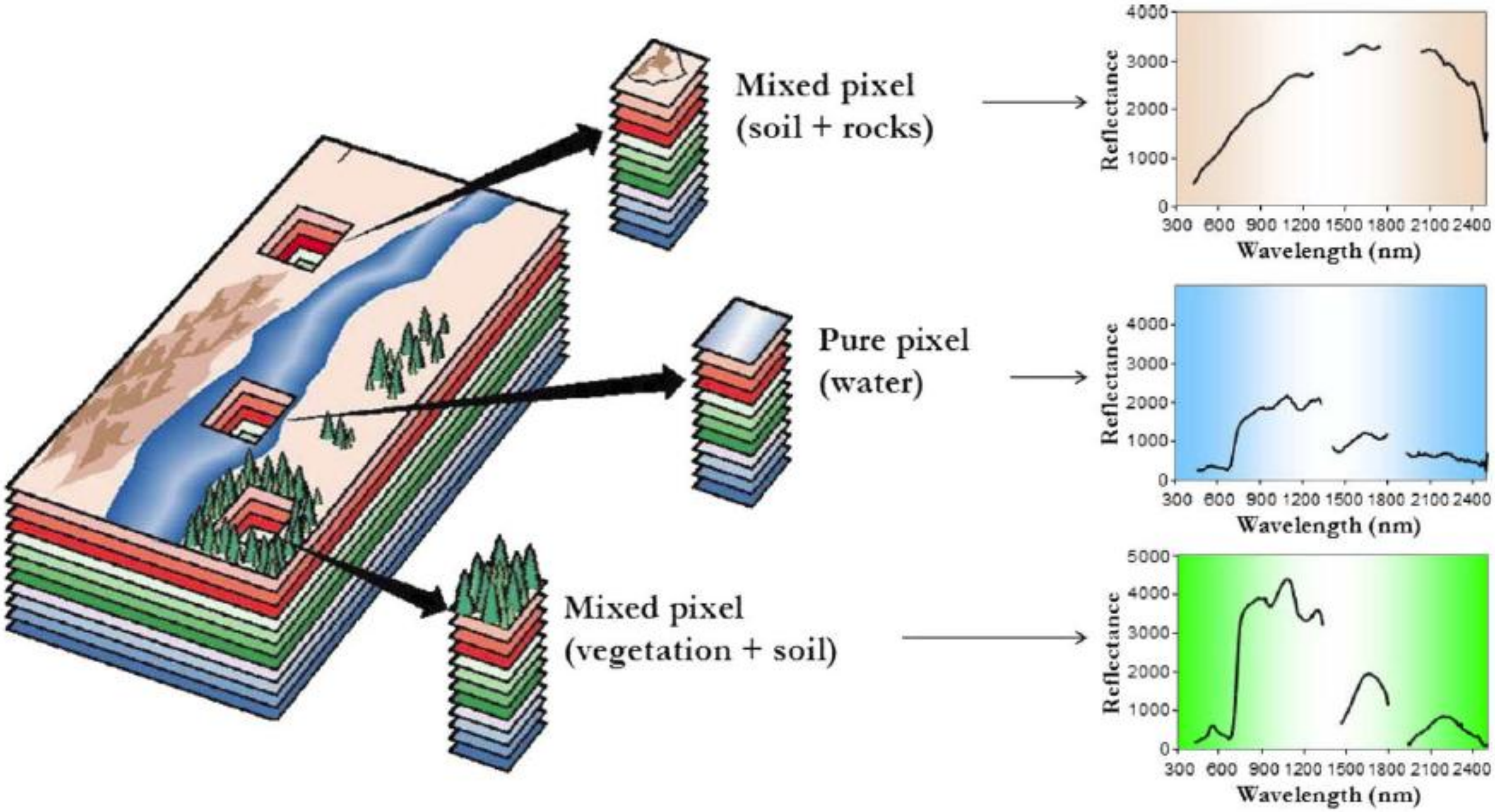
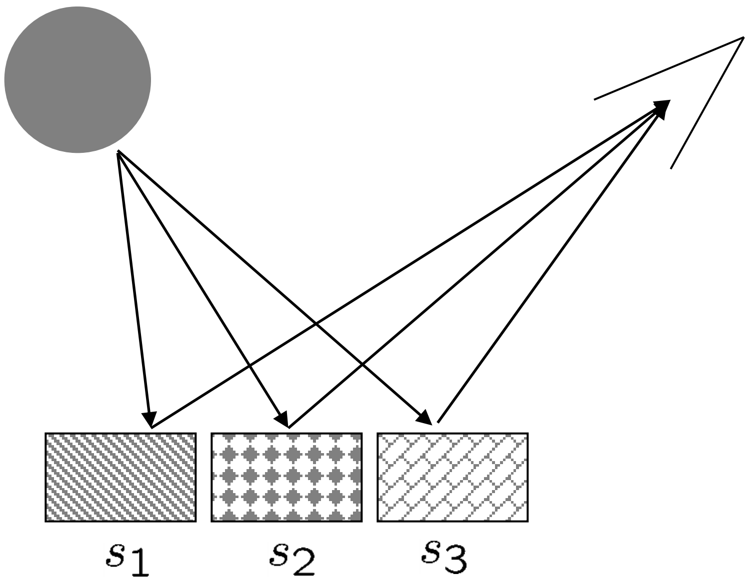
Outline

- Introduction to hyperspectral unmixing
- Hyperspectral unmixing via sparse regression.
- Recovery guarantees/convex and nonconvex solvers
- Improving hyperspectral sparse regression
  - structured sparsity
  - dictionary pruning
- Solving convex sparse hyperspectral unmixing and related convex inverse problems with ADMM
- Concluding remarks

Hyperspectral imaging (and mixing)



Linear mixing model (LMM)



$$\mathbf{y} = \sum_{i=1}^{p} s_i \boldsymbol{\rho}_i \qquad \boldsymbol{\rho}_i = \begin{bmatrix} \rho_{1i} \\ \rho_{2i} \\ \vdots \\ \rho_{mi} \end{bmatrix}$$

$$\mathbf{y} = \mathbf{M}\mathbf{s}$$

$$\mathbf{s} \equiv \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

$$\mathbf{M} \equiv [\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_3]$$

Incident radiation interacts only with one component (checkerboard type scenes)

Hyperspectral linear unmixing → Estimate $\mathbf{M}, \ \mathbf{s}$

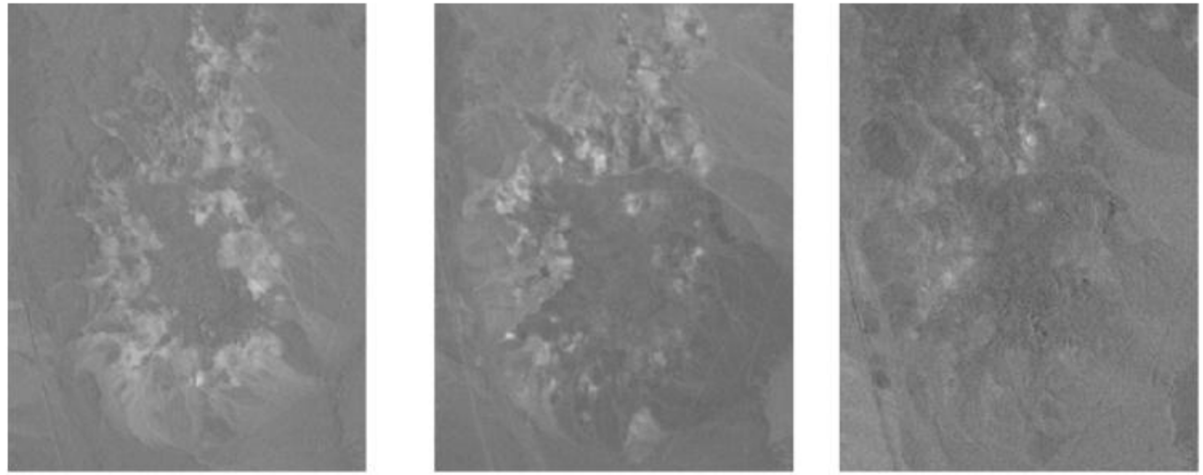
Hyperspectral unmixing

AVIRIS of Cuprite, Nevada, USA

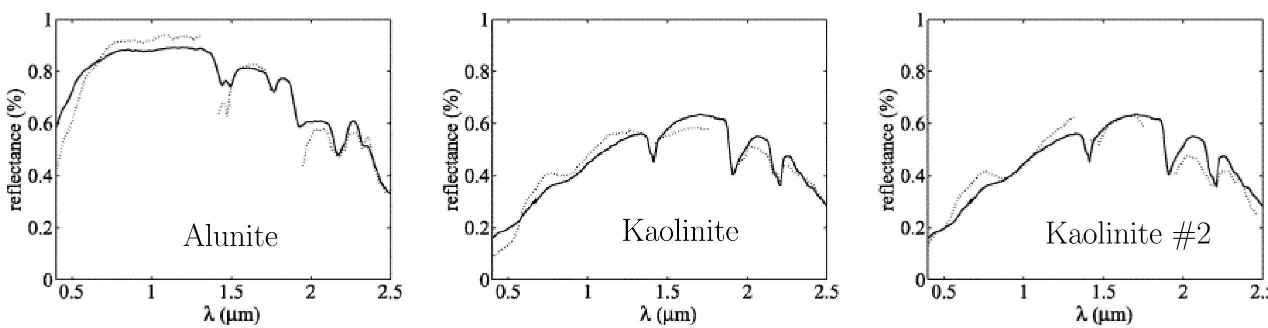
R – ch. 183 (2.10 μm)

G – ch. 193 (2.20 μm)

B – ch. 207 (2.34 μm)



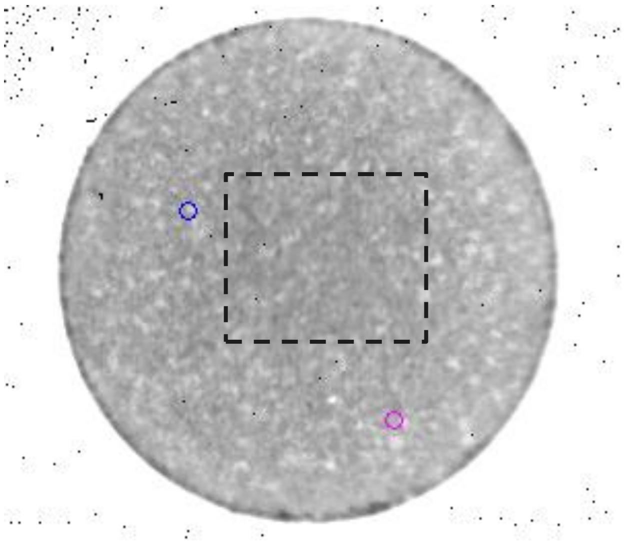
(a) (b) (c)

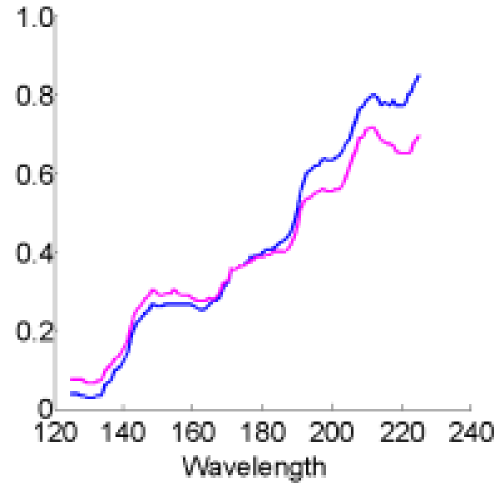


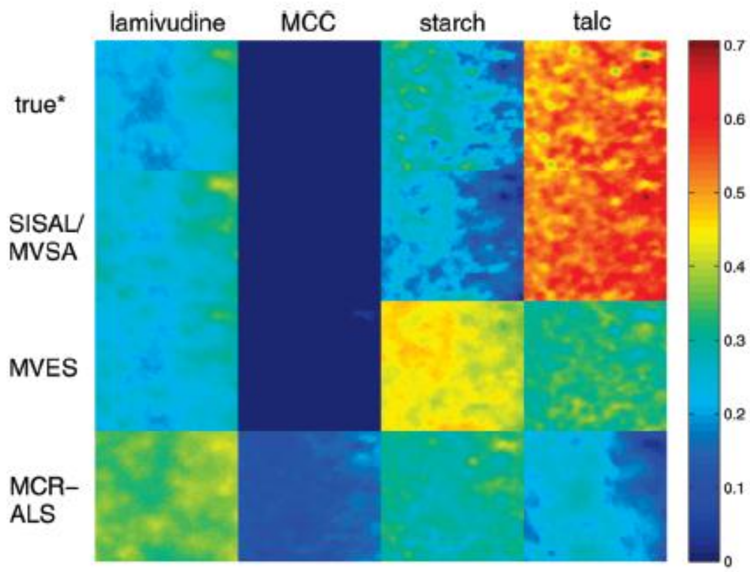
(a) (b) (c)

VCA [Nascimento, B-D, 2005]

NIR tablet imaging







[Lopes et al., 2010]

Spectral linear unmixing (SLU)

Given *N* spectral vectors of dimension *m:* $\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_N]$

Subject to the LMM: $\mathbf{Y} = \mathbf{M}\mathbf{S} + \mathbf{N}$, $\boldsymbol{S} \geq 0$, $\mathbf{1}_p^T \mathbf{S} = \mathbf{1}_N$

ANC: abundance nonnegative constraint

ASC: abundance sum-to-one constraint

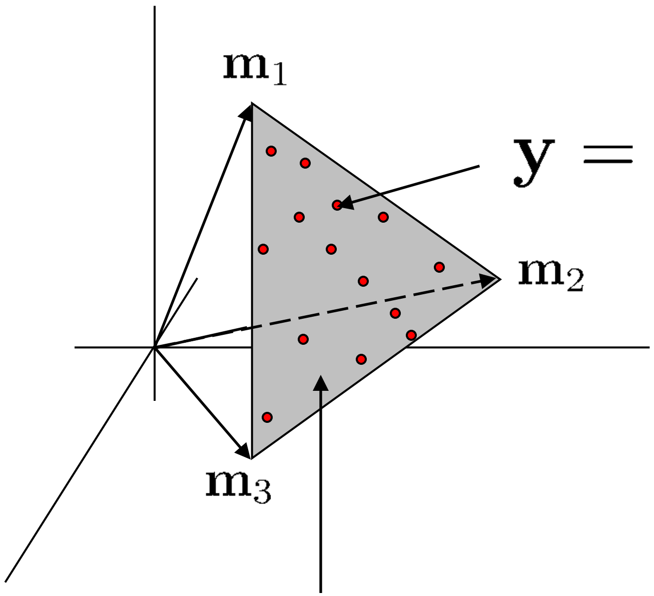
Determine:

- ❑ The mixing matrix $\mathbf{M}$ (*endmember spectra*)
- ❑ The *fractional abundance* vectors $\boldsymbol{S}$,

$\Rightarrow$ SLU is a blind source separation problem (BSS)

Geometrical view of SLU

$$\mathbf{M} = [\mathbf{m}_1, \ldots, \mathbf{m}_p]$$



$$\sum_{j=1}^{p} \alpha_j = 1,\ \alpha_j \geq 0$$

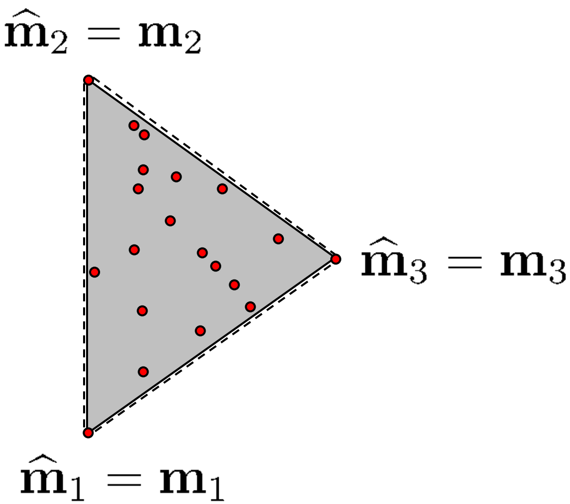
probability simplex (S_I)

$$S_M = \{\mathbf{x} \in \mathbb{R}^p \ :\ \mathbf{x} = \mathbf{M}\boldsymbol{\alpha},\ \boldsymbol{\alpha} \in S_I\}$$ ➡ (p-1) - simplex

Inferring **M** $\Leftrightarrow$ inferring the vertices of the simplex S_M

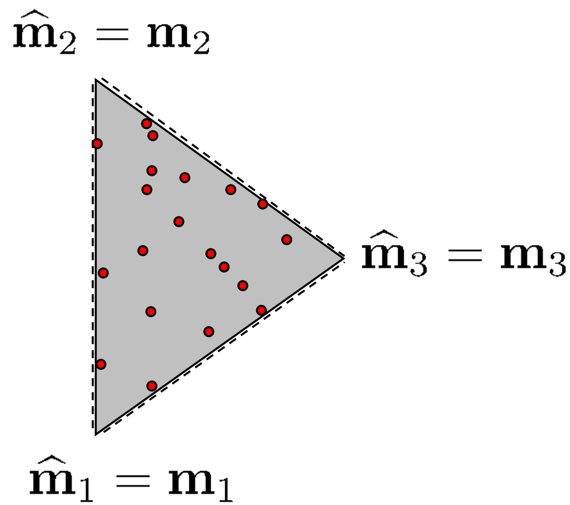
Minimum volume simplex (MVS)

Pure pixels



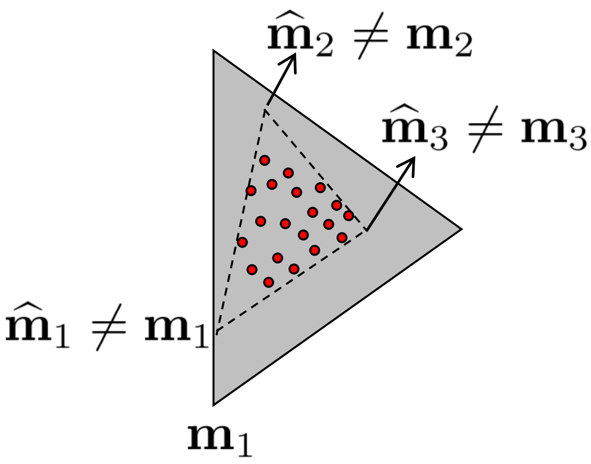
MVS works

No pure pixels



MVS works

No pure pixels



MVS does not work

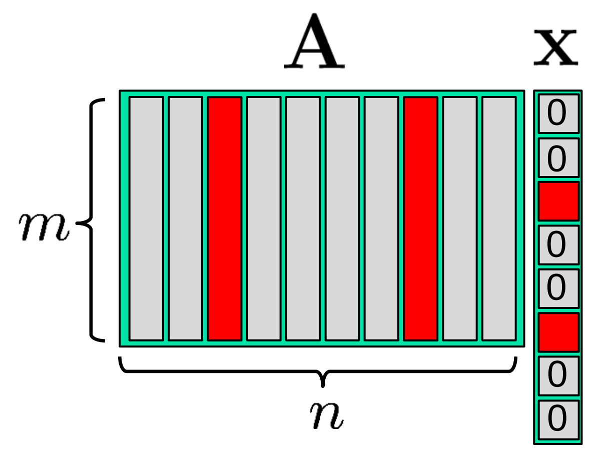
[B-D et al., IEEE JSTATRS, 2012]

Sparse regression-based SLU

- Key observation: Spectral vectors can be expressed as linear combinations of a few pure spectral signatures obtained from a (potentially very large) spectral library

[Iordache, B-D, Plaza, 11, 12]

$$\mathbf{y} = \sum_{i \in S} \mathbf{a}_i \mathbf{x}_i = \mathbf{A}\mathbf{x}$$



- Unmixing: given **y** and **A**, find the sparsest solution of

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

- Advantage: sidesteps endmember estimation

- Disadvantage: **Combinatorial problem !!!**

Sparse reconstruction/compressive sensing

Key result: a sparse signal is exactly recoverable from an underdetermined linear system of equations in a computationally efficient manner via convex/nonconvex programming

[Candes, Romberg, Tao, 06] [Candes,Tao, 06] [Donoho, Tao, 06] [Blumensath, Davies, 09]

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m < n$, and $\mathbf{x}^*$, such that $\mathbf{A}\mathbf{x}^* = \mathbf{y}$

$\mathbf{x}^*$ is the unique solution of $\mathbf{A}\mathbf{x} = \mathbf{y}$ if $2\|\mathbf{x}^*\|_0 < \text{spark}(\mathbf{A})$

$\mathbf{x}^*$ is the solution of the optimization problem

$$(P_0) \quad \min_{\mathbf{x}} \|\mathbf{x}\|_0 \quad \text{subject to } \mathbf{A}\mathbf{x} = \mathbf{y}$$

NP-hard

Sparse reconstruction/compressive sensing

Optimization strategies to cope with P_0 NP-hardness

Convex relaxation

(BP – Basis Pursuit) [Chen et al., 2001]

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ st: } \mathbf{A}\mathbf{x} = \mathbf{y}$$

(BPDN – BP denoising)

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ st: } \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2 \leq \delta$$

(LASSO) [Tibshirani, 1996]

$$\min_{\mathbf{x}} (1/2)\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \lambda\|\mathbf{x}\|_1$$

Approximation algorithms

Bayesian CS [Ji et al., 2008]

CoSaMP – Compressive Sampling Matching Pursuit [Needell, Tropp, 2009]

IHT – Iterative Hard Thresholding [Blumensath, Davies, 09]

GDS - Gradient Descent Sparsification [Garg Khandekar, 2009]

HTP – Hard Thresholding Pursuit [Foucart, 10]

MP - Message Passing [Villa Schniter, 2012]

Exact recovery of sparse vectors

Recovery guarantees: linked with the restricted isometric property (RIP)

Restricted isometric constant: $\delta_p(\mathbf{A}), \quad \mathbf{A} \in \mathbb{R}^{m \times n}$:

$$(1-\delta)\|\mathbf{x}\|_2 \leq \|\mathbf{A}\mathbf{x}\| \leq (1+\delta)\|\mathbf{x}\|_2, \quad \|\mathbf{x}\|_0 \leq p$$

Many SR algorithms ensure exact recovery provided that:

$$\delta_t(\mathbf{A}) \leq \delta_*, \text{ for some t and } \delta_*$$

This condition is satisfied for random matrices provided that

$$m \simeq c\frac{t}{\delta_*^2}\log(n/t)$$

Algorithm	BP	HTP	CoSaMP	GDP	IHT
$\delta_t < \delta_*$	$\delta_{2s} < 0.465$	$\delta_{3s} < 0.577$	$\delta_{4s} < 0.384$	$\delta_{2s} < 0.333$	$\delta_{3s} < 0.555$
Ratio t/δ_*^2	9.243	**9**	27.08	18	12

(from [Foucart, 10])

Example: Gaussian matrices; signed signals

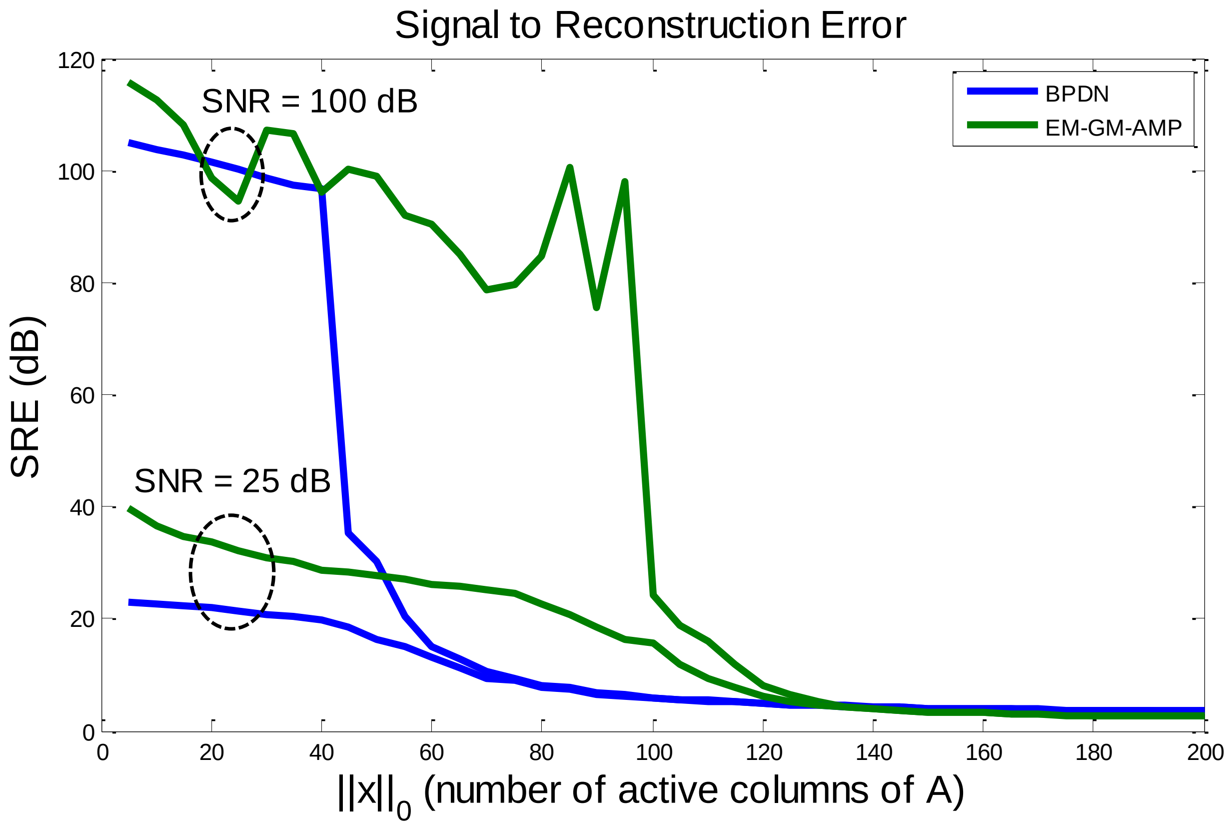
$$\mathbf{A} \in \mathbb{R}^{m\times n}; \quad (m = 200, n = 400); \quad \mathcal{N}(0,1); \text{ iid}; \quad x_i \sim \mathcal{N}(0,1)$$

Algorithms:

(BPDN – SUnSAL) [B-D, Figueiredo, 2010]

$$\min_{\mathbf{x}} \ \|\mathbf{x}\|_1 \ \text{ st: } \ \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2 \leq \delta$$

EM-BG-AMP
[Villa Schniter, 2012]



Example: Gaussian matrices; non-negative signals

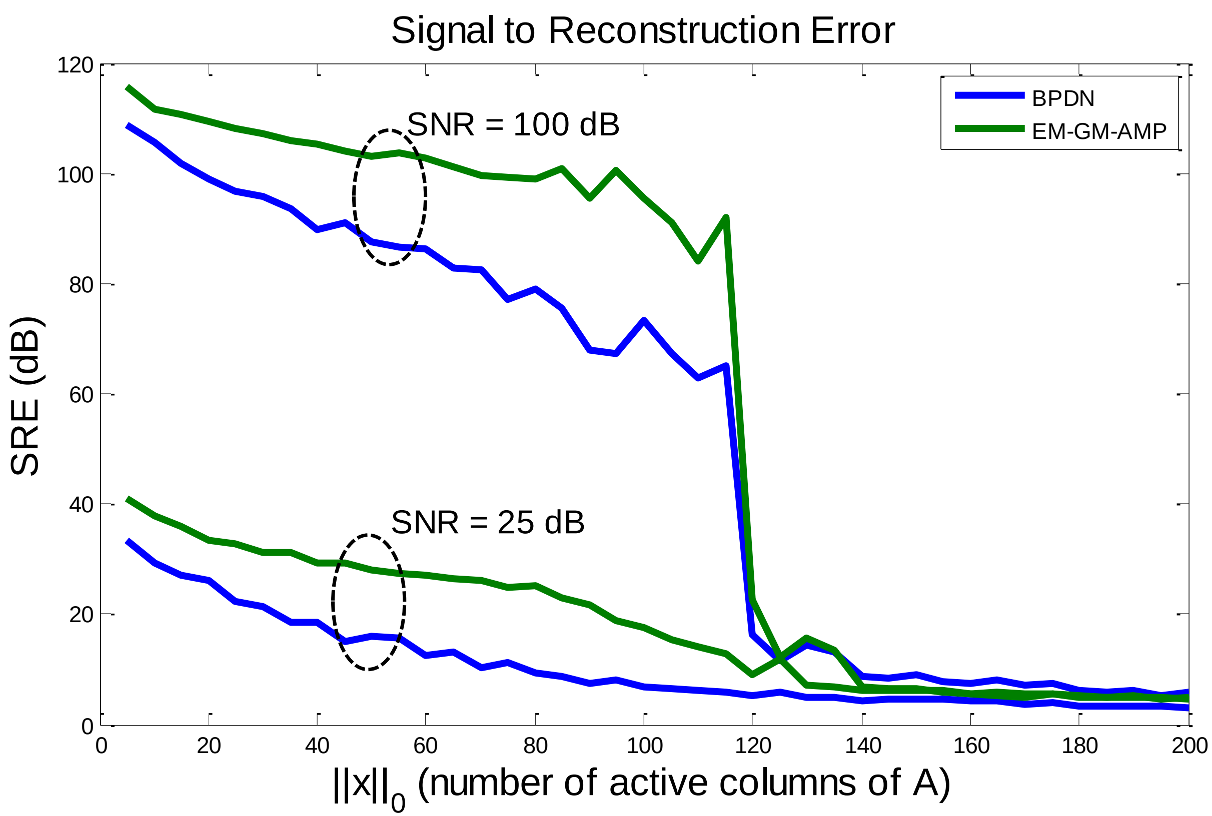
$\mathbf{A} \in \mathbb{R}^{m \times n}$; $\mathcal{N}(0,1)$ iid ; $m = 200, n = 400$; $x_i \sim$ UD in the simplex

Algorithms:

(BPDN – SUnSAL)

$$\min_{\mathbf{x}} \ \|\mathbf{x}\|_1 \ \text{ st: } \ \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2 \le \delta, \ \mathbf{x} \ge \mathbf{0}$$

EM-BG-AMP



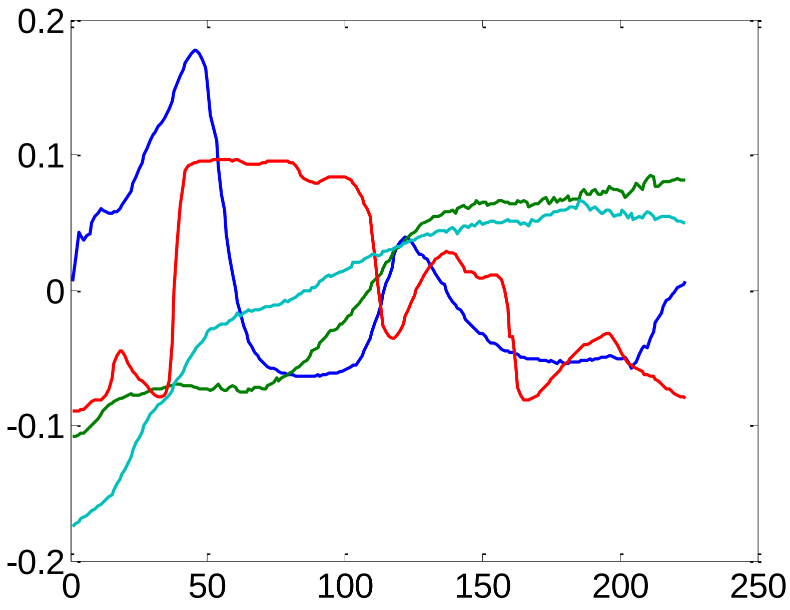
Hyperspectral libraries

Hyperspectral libraries exhibit poor RI constants

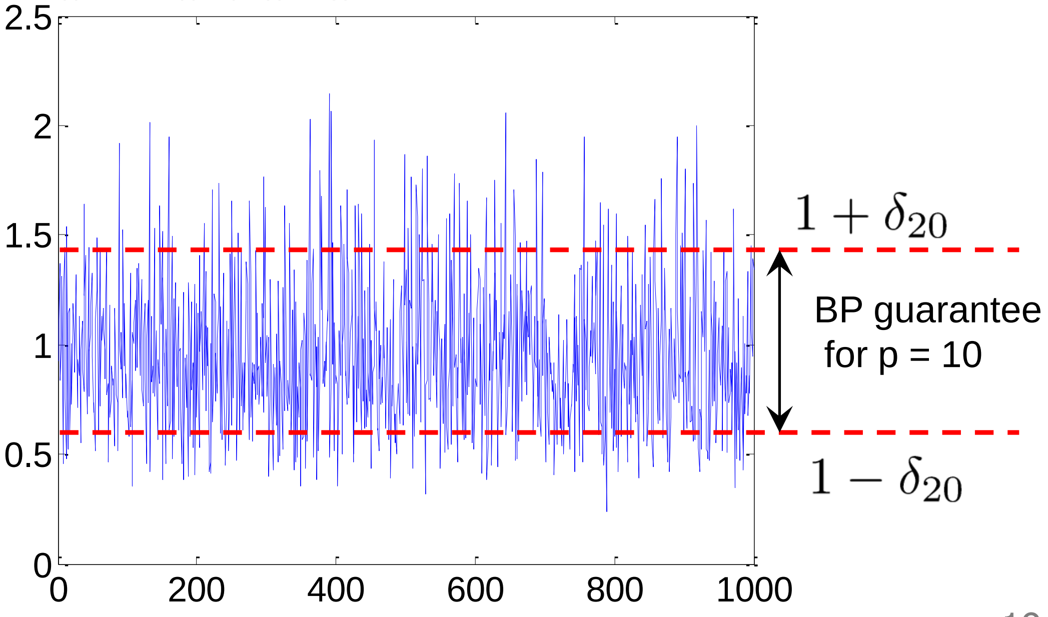
(Mutal coherence close to 1 [Iordache, B-D, Plaza, 11, 12])

Illustration: $\mathbf{A} \in \mathbb{R}^{m \times n}$; subset from USGS; $m = 200, n = 400$

Normalized and mean removed



$\|\mathbf{A}\mathbf{x}\|_2/\|\mathbf{x}\|_2$ $\|\mathbf{x}\|_0 = 10$



Example: Hyperspectral library; non-negative signals

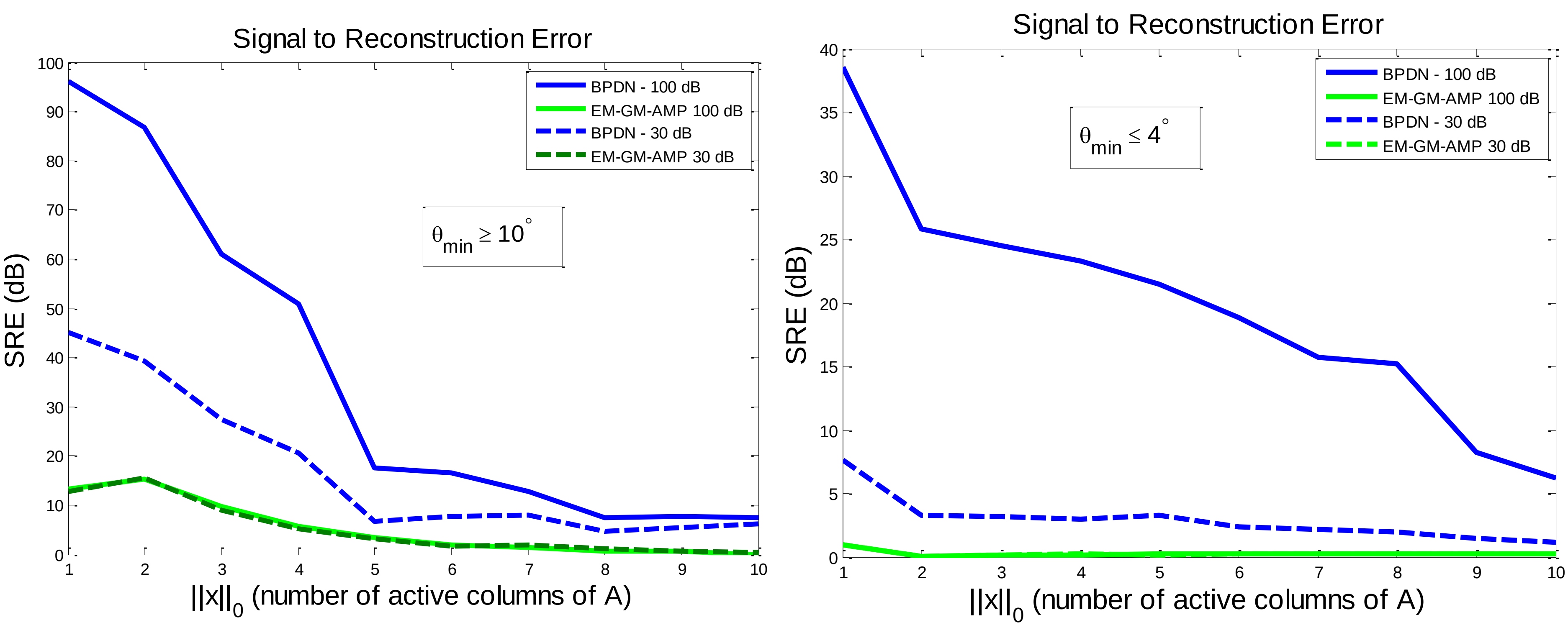
Illustration: $\mathbf{A} \in \mathbb{R}^{m \times n}$; subset from USGS; $m = 200, n = 300$

Algorithms:

(BPDN – SUnSAL)

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ st: } \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2 \leq \delta, \ \mathbf{x} \geq \mathbf{0}$$

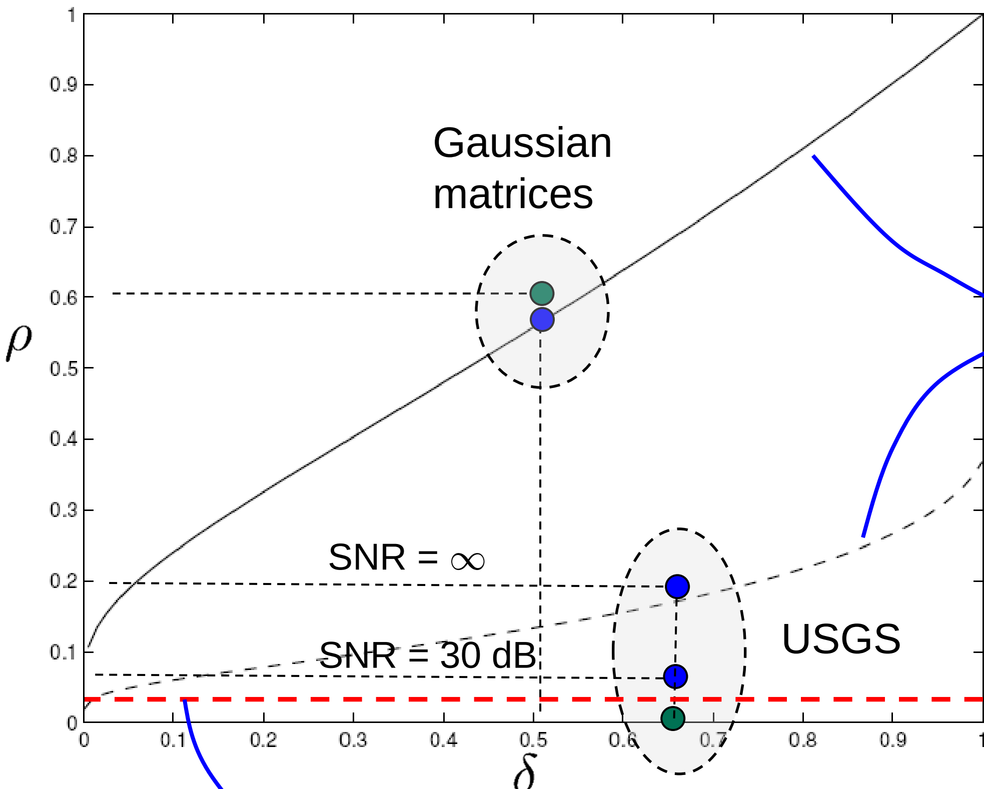
EM-BG-AMP



Phase transition curves

$\ell_1, -/\ell_0$ equivalence for non-negative signals $(\mathbf{A} \in \mathbb{R}^{m \times n})$

$\delta = m/n$ undersamplig factor $\rho = \|\mathbf{x}\|_0/m$ fractional sparsity



EM-BG-AMP

BPDN

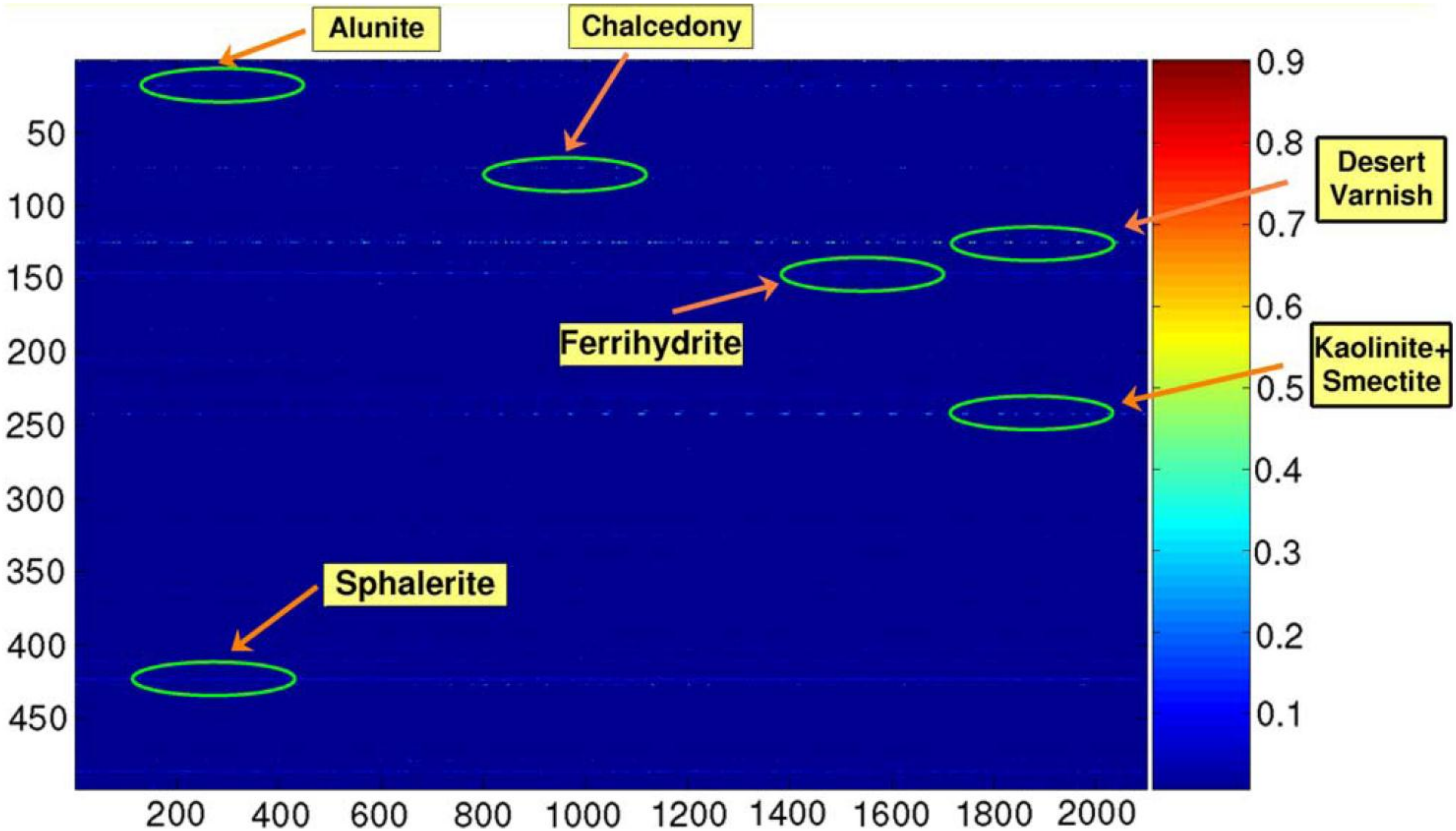
$\mathbf{A}$ - orthogonal projector

Results linked with the k-neighborliness property of polytopes

[Donoho, Tanner, 2005]

$\rho = 0.025$ (enough for many hyperspectral applications)

Real data – AVIRIS Cuprite



[Iordache, B-D, Plaza, 11, 12]

Sparse reconstruction of hyperspectral data: Summary

Bad news: Hyperspectral libraries have poor RI constants

Good news: Hyperspectral mixtures are highly sparse, very often $p \leq 5$

Surprising fact: Convex programs (BP, BPDN, LASSO, …) yield much better empirical performance than non-convex state-of-the-art competitors

Directions to improve hyperspectral sparse reconstruction

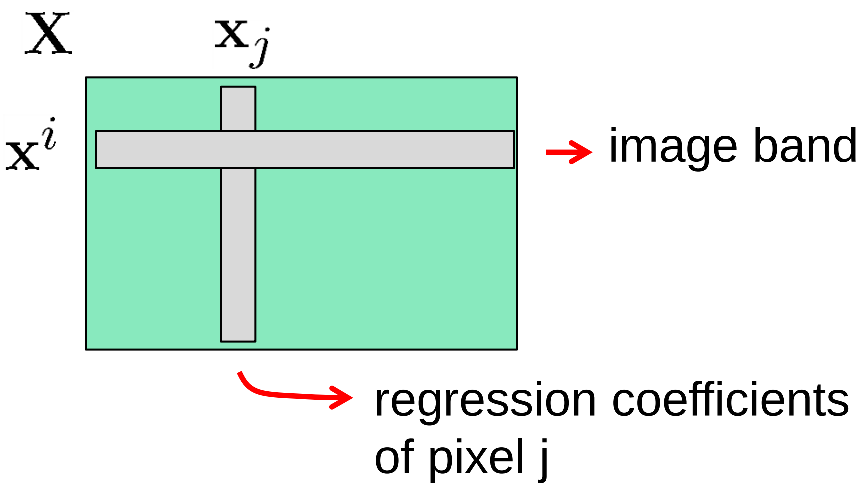
- Structured sparsity + subspace structure
  (pixels in a give data set share the same support)
- Spatial contextual information
  (pixels belong to an image)

Beyond l_1 pixelwise regularization

Rationale: introduce new sparsity-inducing regularizers to counter the sparse regression limits imposed by the high coherence of the hyperspectral libraries.

Let's rewrite the LMM as $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{N}$

$$\begin{cases} \mathbf{Y} \in \mathbb{R}^{m\times N} \\ \mathbf{A} \in \mathbb{R}^{m\times n} \\ \mathbf{X} \in \mathbb{R}^{n\times N} \end{cases}$$



Constrained total variation sparse regression (CTVSR)

$$\min_{\mathbf{X}} (1/2)\|\mathbf{AX}-\mathbf{Y}\|_F^2 + \lambda_1\|\mathbf{X}\|_1 + \lambda_2\phi_{TV}(\mathbf{X})$$

subject to: $\mathbf{X} \geq \mathbf{0}$

[Iordache, B-D, Plaza, 11]

total Variation of i-th band

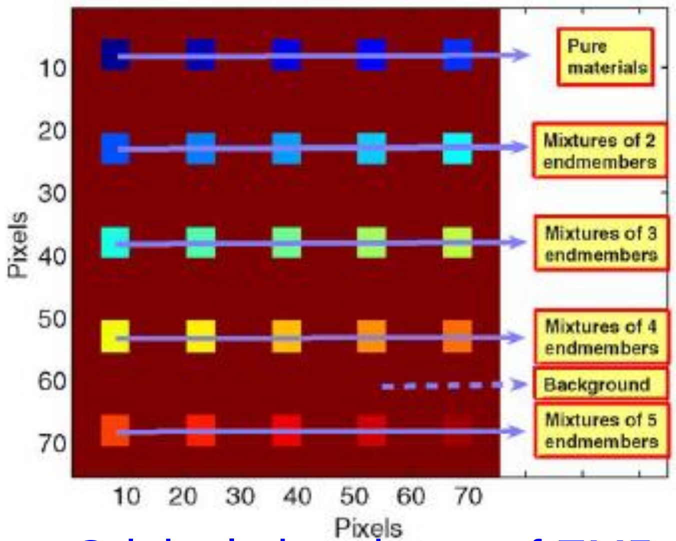
$$\phi_{TV}(\mathbf{X}) := \sum_{i=i}^{n} \|\mathbf{L}\mathbf{x}^i\|_1 = \sum_{i=1}^{n}\sum_{k=1}^{N}\sqrt{([\mathbf{D}_h\mathbf{x}^i]_k)^2 + ([\mathbf{D}_v\mathbf{x}^i]_k)^2}$$

Related work [Zhao, Wang, Huang, Ng, Plemmons, 12]

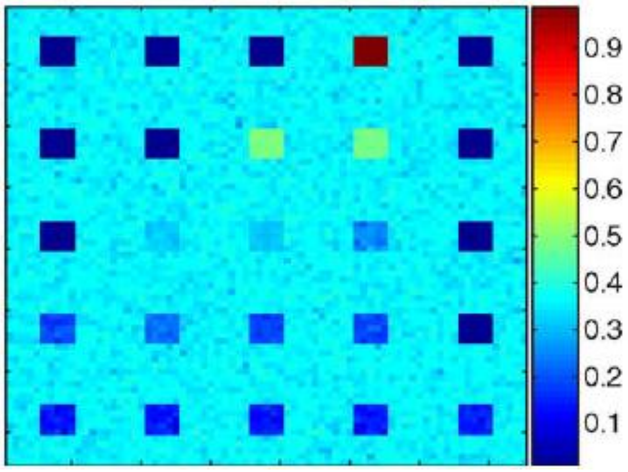
Ilustrative examples with simulated data : SUnSAL-TV

$\mathbf{A} \in \mathbb{R}^{224\times 240}$ (from USGS library) $\quad (m = 224, N = 75 \times 75, k = 5)$

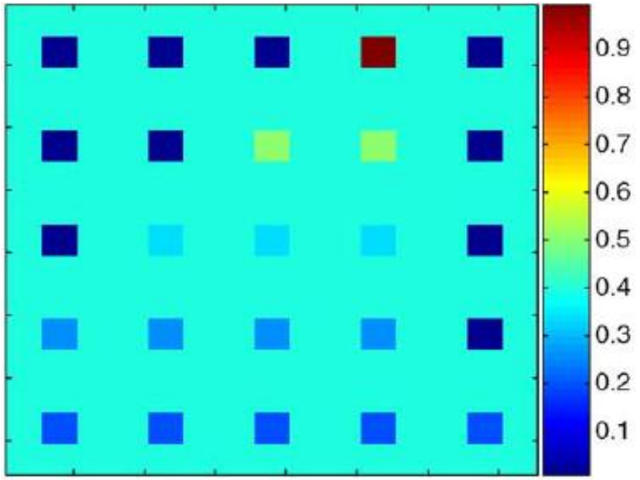
Original data cube



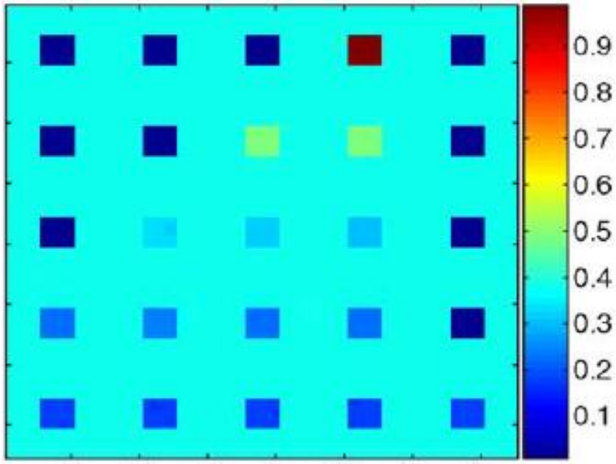
SUnSAL estimate



Original abundance of EM5



SUnSAL-TV estimate



Constrained colaborative sparse regression (CCSR)

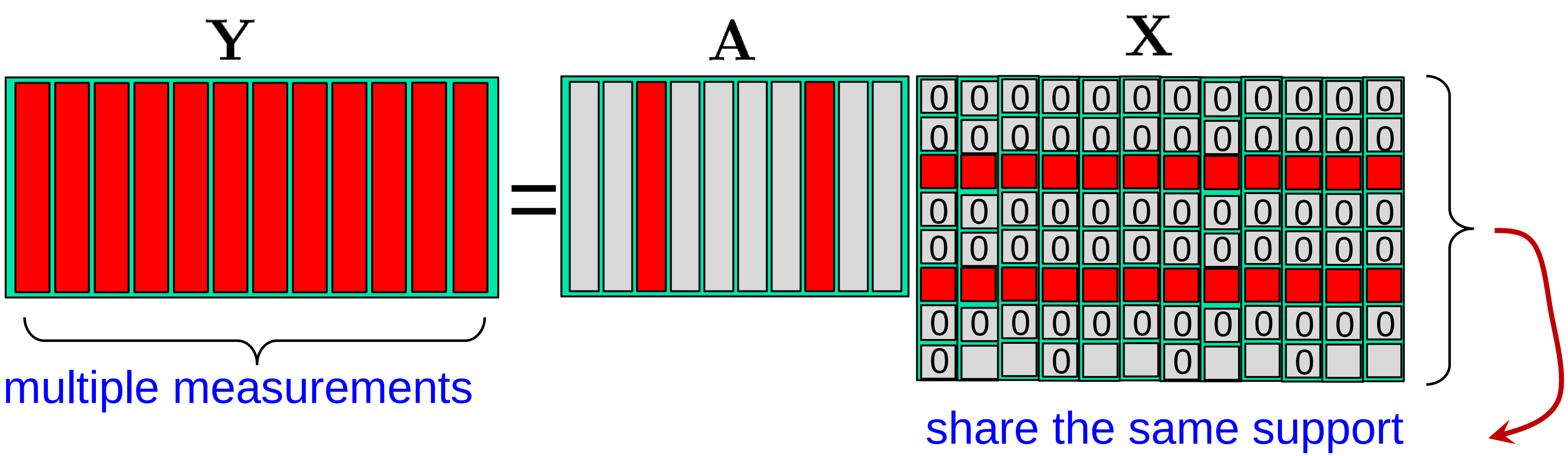
$$\min_{\mathbf{X}} \ (1/2)\|\mathbf{A}\mathbf{X} - \mathbf{Y}\|_F^2 + \lambda\|\mathbf{X}\|_{2,1}$$

$$\|\mathbf{X}\|_{2,1} := \sum_{i=1}^{n} \|\mathbf{x}^i\|_2$$

$$\text{subject to: } \ \mathbf{X} \geq \mathbf{0}, \quad \mathbf{1}_n^T\mathbf{X} = \mathbf{1}_N^T$$

[Iordache, B-D, Plaza, 11, 12]

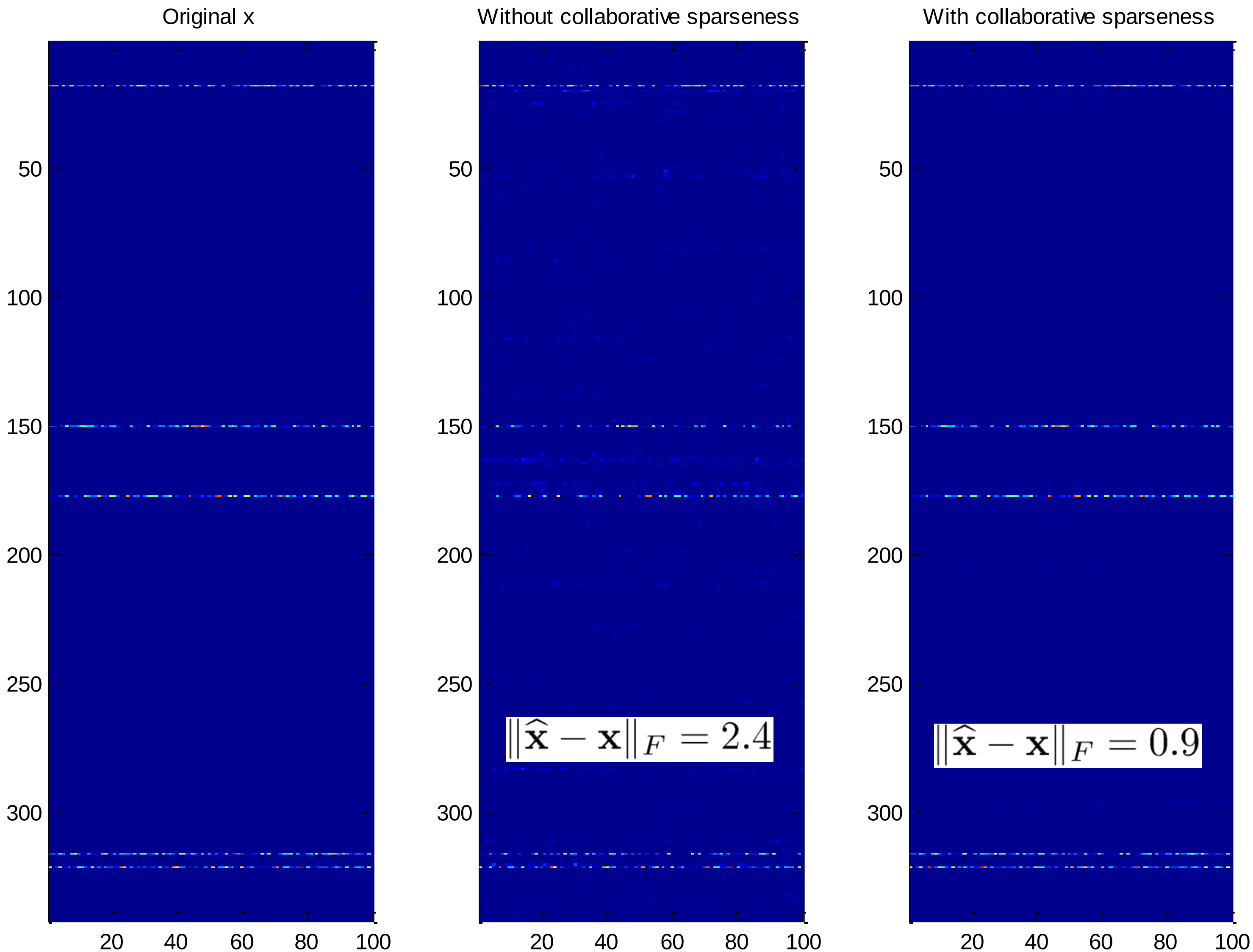
[Turlach, Venables, Wright, 2004]



Theoretical guaranties (superiority of multichanel) : the probability of recovery failure decays exponentially in the number of channels. [Eldar, Rauhut, 11]

Ilustrative examples with simulated data : CSUnSAL

$\mathbf{A} \in \mathbb{R}^{224\times 350}$ (from USGS library) $\mathbf{x} \in \mathbb{R}^{350\times 100}$ (sparsity k = 5)



SNR = 35dB

time = 10 sec

Multiple measurements

The multiple measurement vector (MMV) problem

$$\text{minimize } \|\mathbf{X}\|_0$$

$$\text{subject to: } \mathbf{Y} = \mathbf{A}\mathbf{X}$$

$\|\mathbf{X}\|_0$ - number of non-null rows of $\mathbf{X}$

MMV has a unique solution iff

[Feng, 1997], [Chen, Huo, 2006], [Davies, Eldar, 2012]

$$\|\mathbf{X}\|_0 < \frac{\text{spark}(\mathbf{A}) + \text{rank}(\mathbf{Y}) - 1}{2}$$

MMV gain

If $\text{rank}(\mathbf{Y}) = \|\mathbf{X}\|_0$, the above bound is achieved using the multiple sinal classification (MUSIC) algorithm

Endemember identification with MUSIC

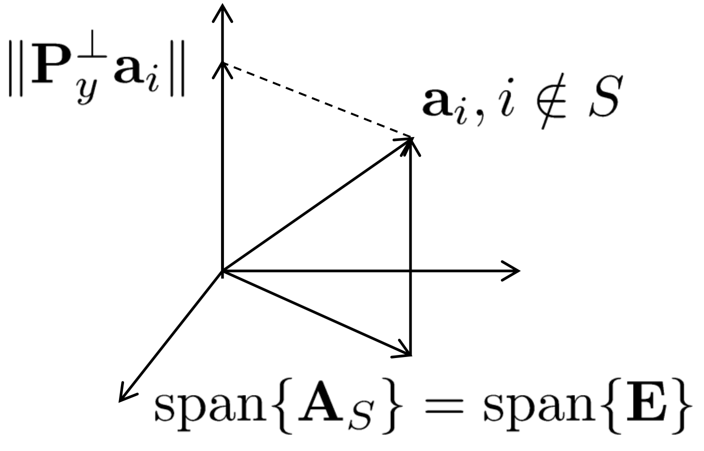
$\mathbf{Y} = \mathbf{A}\mathbf{X}$ (noiseless measurements)

MUSIC algorithm

1) Compute $\mathbf{E} = [\mathbf{e}_1, \ldots, \mathbf{e}_p]$, the first *p* eigenvalues of $\mathbf{R}_y = \mathbf{Y}\mathbf{Y}^T/n$

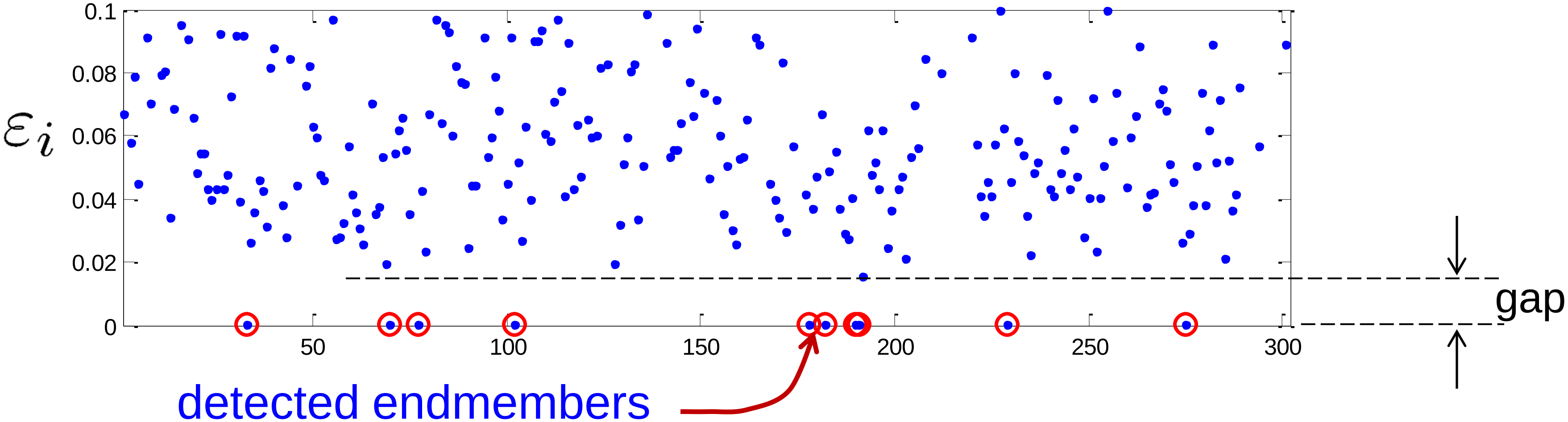
2) Compute $\varepsilon_i = \dfrac{\|\mathbf{P}_y^{\perp}\mathbf{a}_i\|}{\|\mathbf{a}_i\|}$, for $i = 1, \ldots, m$ and set

$\mathbf{M} = \mathbf{A}_S$ with $S = \{i \,:\, \varepsilon_i = 0, i = 1, \ldots, m\}$



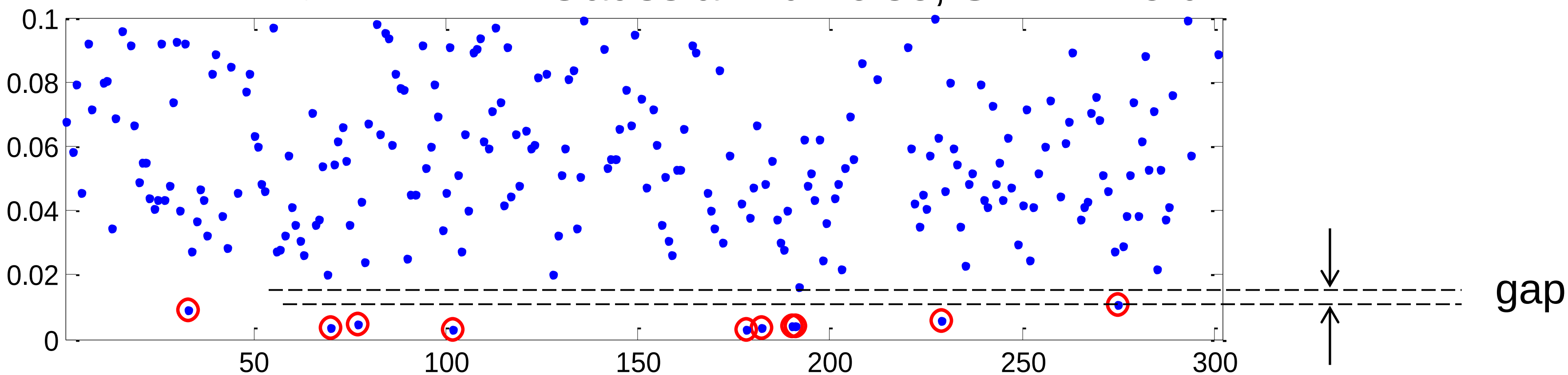
Examples (simulated data)

A – USGS ($\geq 3^\circ$) p = 10, N=5000, X - uniform over the simplex, SNR = ∞,



$$\mathbf{Y} = \mathbf{AX} + \mathbf{N}$$

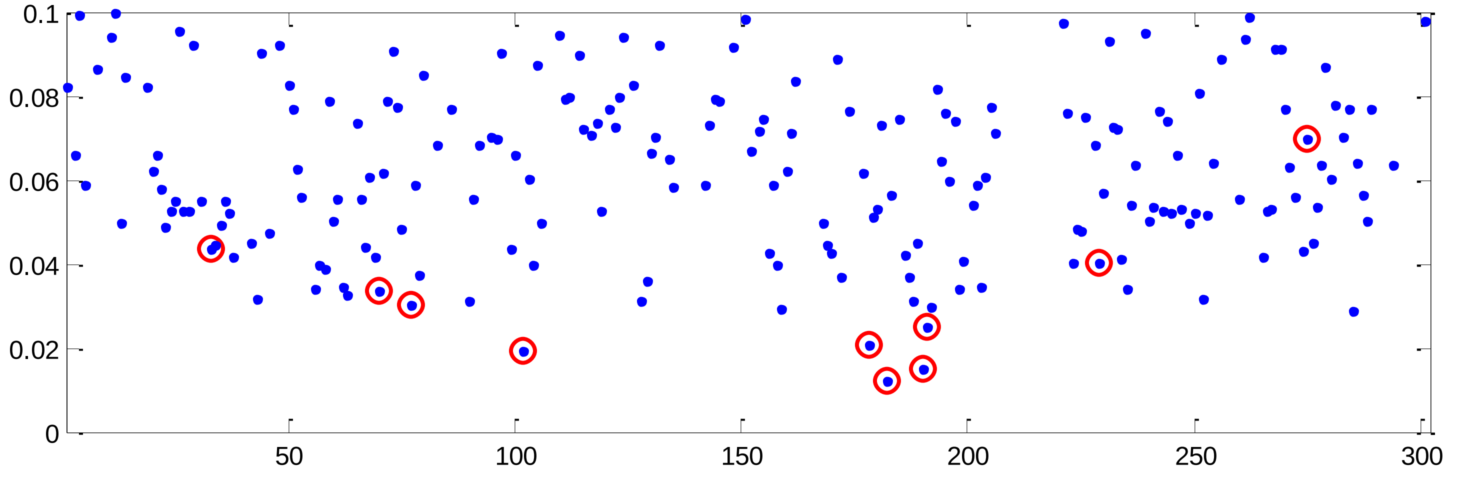
Gaussian iid noise, SNR = 25 dB



Examples (simulated data)

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{N}$$

colored noise, SNR = 25 dB



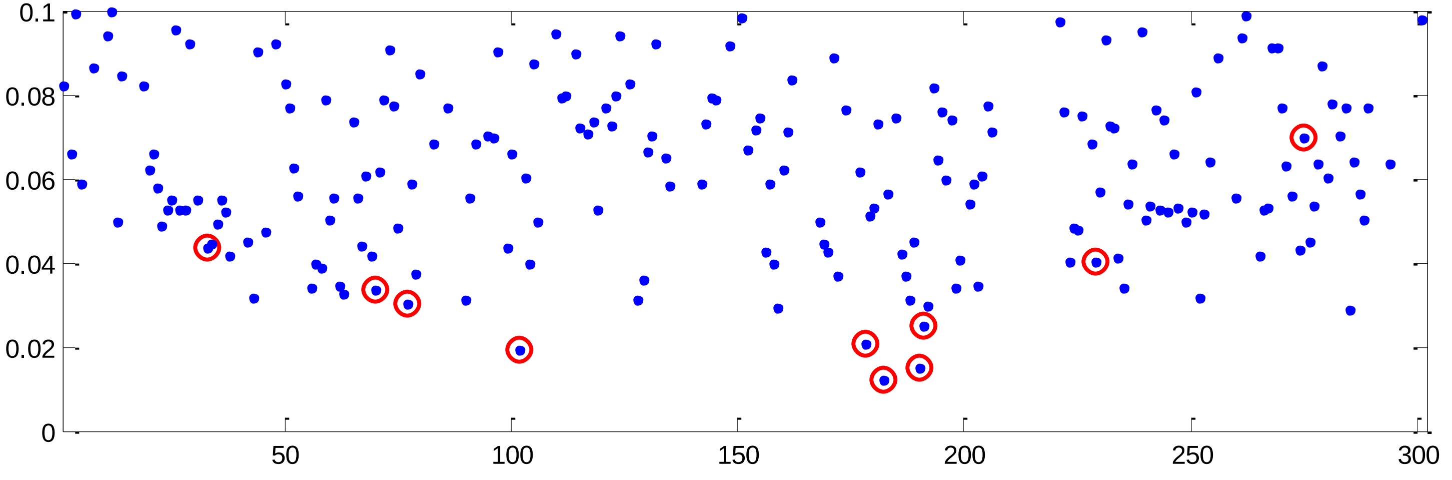
cause of the large projection errors: poor identification of the subspace signal

$$\mathbf{R}_y \simeq \mathbf{A}_S \mathbf{R}_x \mathbf{A}_S^T + \mathbf{R}_n \quad (\mathbf{R}_n \neq \sigma^2 \mathbf{I}) \quad \Rightarrow \text{span}\{\mathbf{E}\} \neq \text{span}\{\mathbf{A}_S\}$$

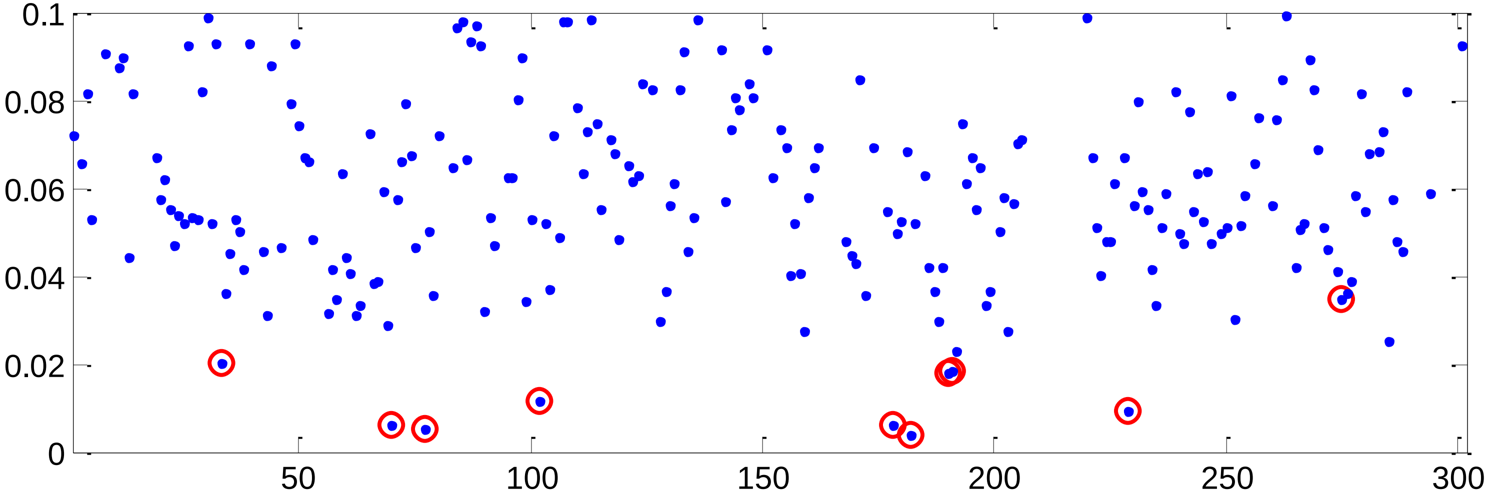
cure: identify the signal subspace

Signal subspace identification

colored noise, SNR = 25 dB (incorrect signal subspace)



colored noise, SNR = 25 dB (signal subspace identified with HySime, [BD, Nascimento, 2008])



Proposed MUSIC – Colaborative SR algorithm

MUSIC-CSR algorithm [B-D, 2012]

1) Estimate the signal subspace $\mathrm{span}\{\mathbf{A}_S\}$ using, e.g. the HySime algorithm.

2) Compute $\varepsilon_i = \frac{\|\mathbf{P}_y^{\perp}\mathbf{a}_i\|}{\|\mathbf{a}_i\|}$, for $i = 1, \ldots, m$ and define the index set $S = [i \, : \, \varepsilon_i \leq \delta, i = 1, \ldots, m]$

3) Solve the colaborative sparse regression optimization

$$\min_{\mathbf{X}} (1/2)\|\mathbf{Y} - \mathbf{A_S}\mathbf{X}\|^2 + \lambda\|\mathbf{X}\|_{2,1}, \quad \mathbf{X} \geq 0$$

[B-D, Figueiredo, 2012]

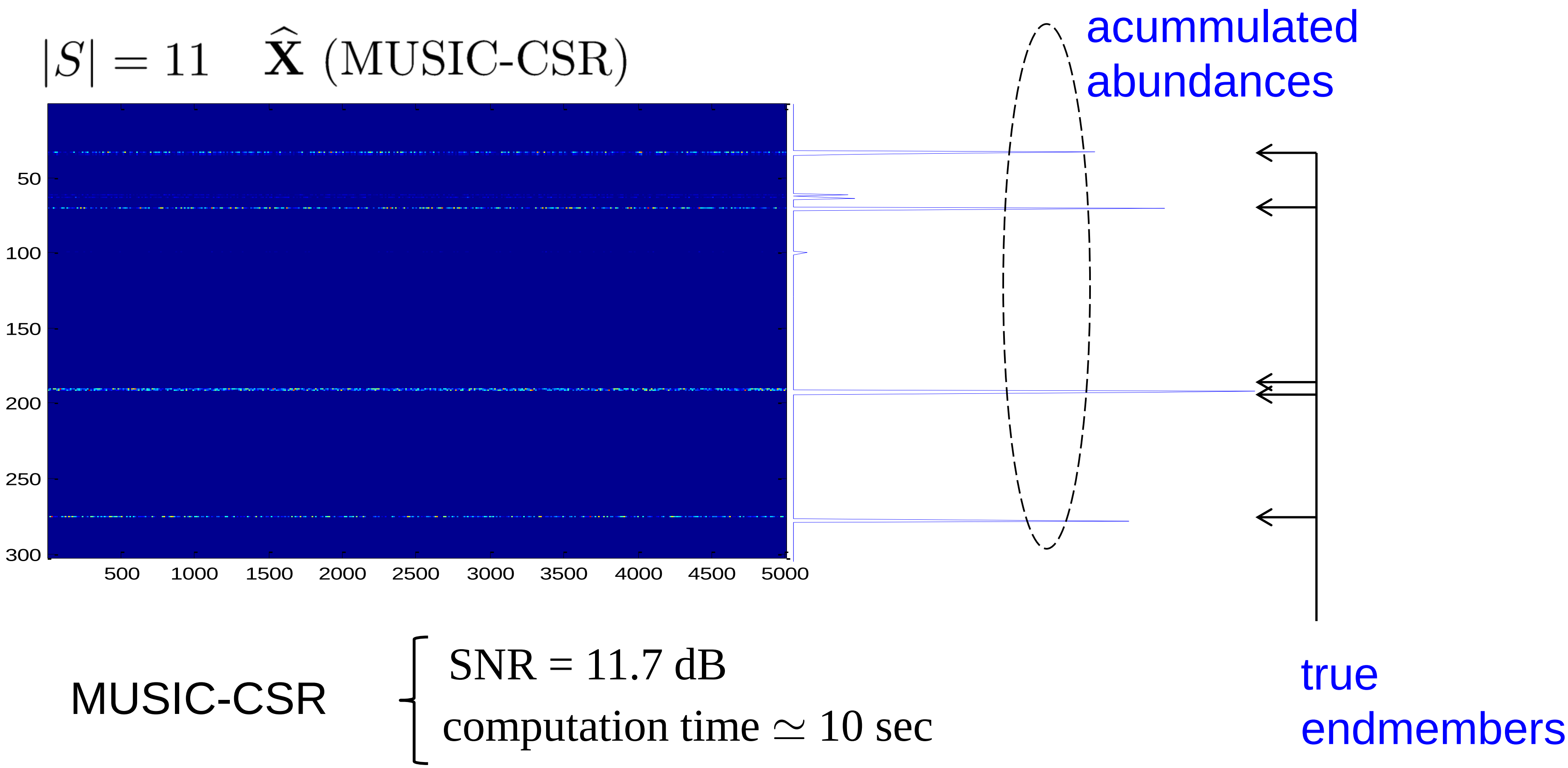
Related work: CS-MUSIC [Kim, Lee, Ye, 2012]

(N < k and iid noise)

MUSIC – CSR results

A – USGS ($\geq 3^\circ$), Gaussian shaped noise, SNR = 25 dB, k = 5, m = 300,

$|S| = 11 \quad \widehat{\mathbf{X}}$ (MUSIC-CSR)

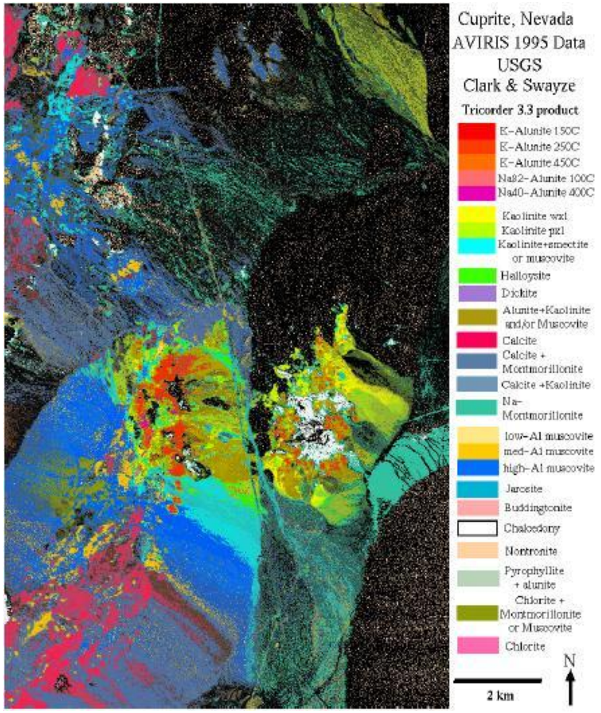


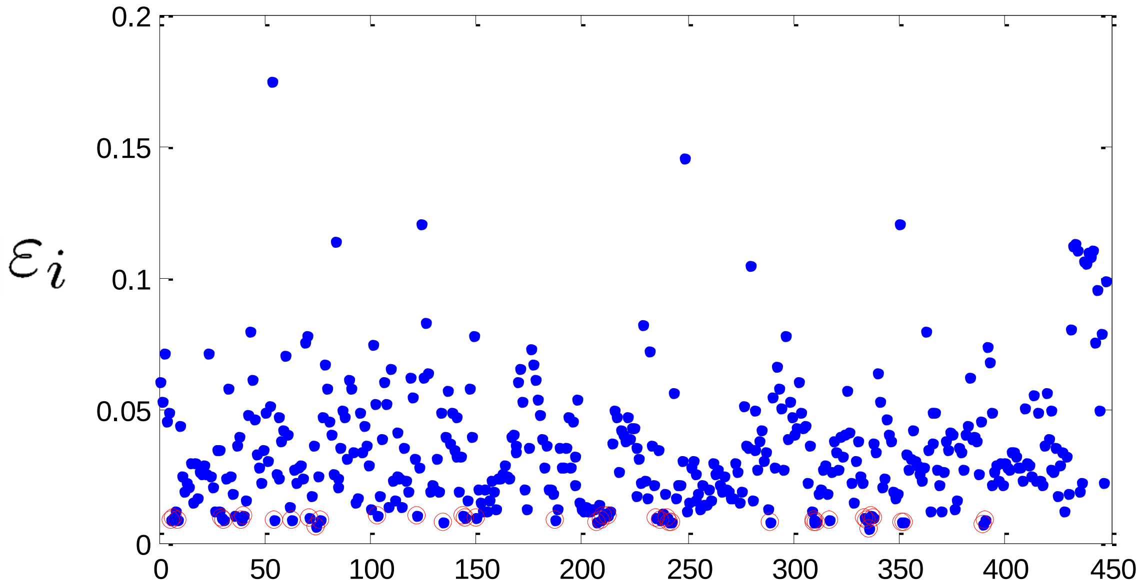
MUSIC-CSR $\begin{cases} \text{SNR} = 11.7 \text{ dB} \\ \text{computation time} \simeq 10 \text{ sec} \end{cases}$

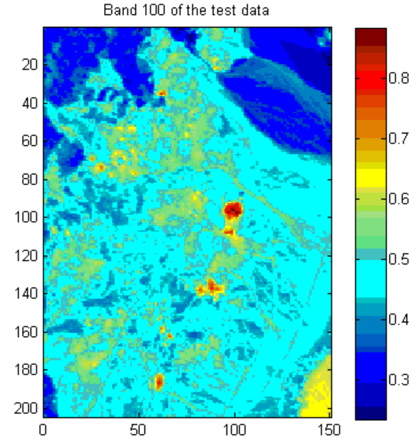
CSR $\begin{cases} \text{SNR} = 0 \text{ dB} \\ \text{computation time} \simeq 600 \text{ sec} \end{cases}$

Results with CUPRITE

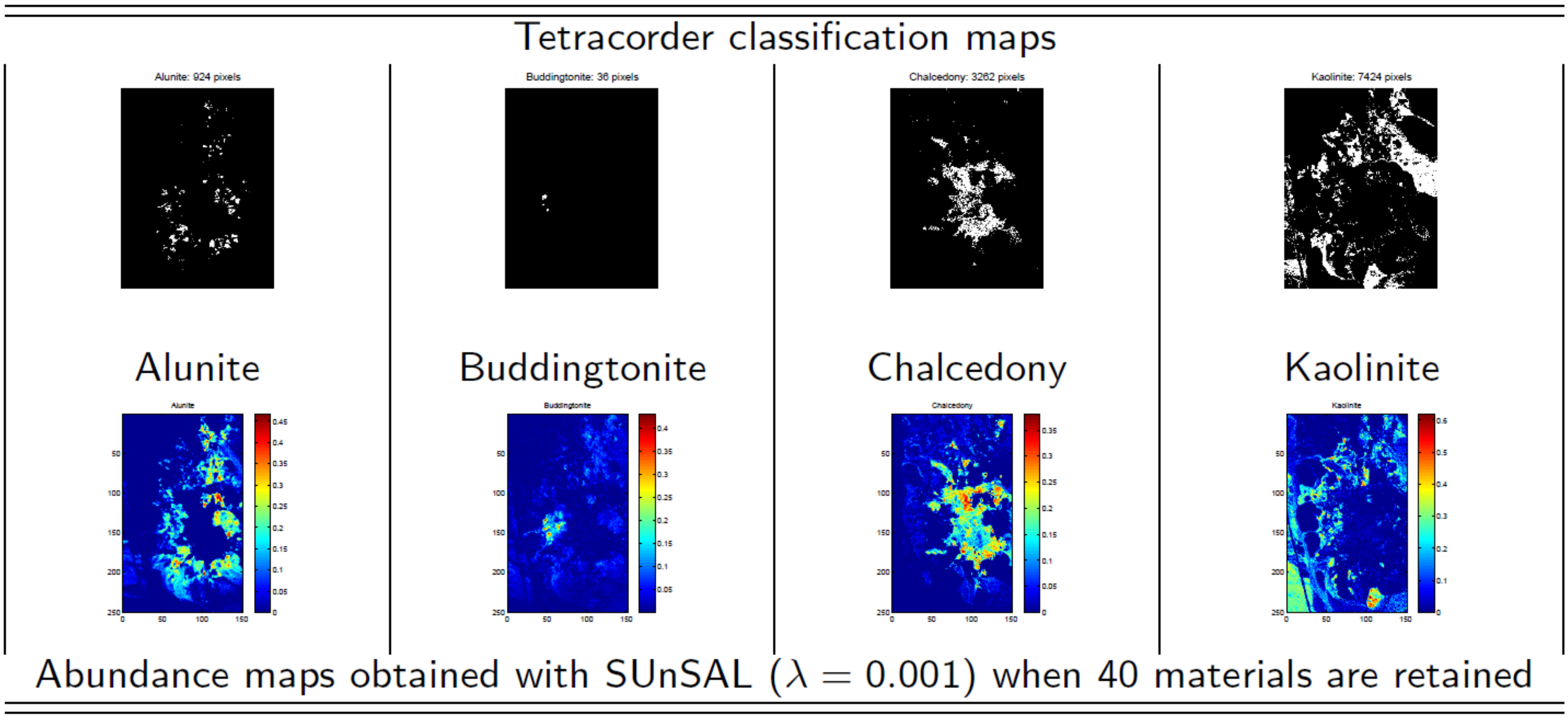
- size: 350x350 pixels
- spectral library: 302 materials (minerals) from the USGS library
- spectral bands: 188 out of 224
  (noisy bands were removed)
- spectral range: 0.4 – 2.5 um
- spectral resolution: 10 nm
- “validation” map: Tetracorder***







Results with real data



Note: Good spatial distribution of the endmembers

Processing times: 2.6 ms/pixel using the full library; 0.22ms/pixels using the pruned library with 40 members

Convex optimization problems in SLU

Constrained least squares (CLS)

$$\min_{\mathbf{X}} \ \|\mathbf{AX}-\mathbf{Y}\|_F^2$$
$$\text{subject to: } \ \mathbf{X} \geq \mathbf{0}$$

$$\|\mathbf{X}\|_F^2 := \operatorname{tr}\{\mathbf{X}\mathbf{X}^T\}$$

Fully constrained least squares (FCLS)

$$\min_{\mathbf{X}} \ \|\mathbf{AX}-\mathbf{Y}\|_F^2$$
$$\text{subject to: } \ \mathbf{X} \geq \mathbf{0}, \ \mathbf{1}_n^T\mathbf{X} = \mathbf{1}_N^T$$

$\mathbf{X} \geq \mathbf{0}$ → ANC

$\mathbf{1}_n^T\mathbf{X} = \mathbf{1}_N^T$ → ASC

Constrained sparse regression (CSR)

$$\min_{\mathbf{X}} \ (1/2)\|\mathbf{AX}-\mathbf{Y}\|_F^2 + \lambda\|\mathbf{X}\|_1$$
$$\text{subject to: } \ \mathbf{X} \geq \mathbf{0}$$

$$\|\mathbf{X}\|_1 := \sum_{i=1}^N \|\mathbf{x}_i\|_1$$

Convex optimization problems in SLU

Constrained basis pursuit (CBP)

$$\min_{\mathbf{X}} \ \|\mathbf{X}\|_1$$
$$\text{subject to: } \ \mathbf{AX} = \mathbf{Y}, \ \ \mathbf{X} \geq \mathbf{0}$$

CBP denoising (CBPDN)

$$\min_{\mathbf{X}} \ \|\mathbf{X}\|_1$$
$$\text{subject to: } \ \|\mathbf{AX} - \mathbf{Y}\|_F \leq \delta, \ \ \mathbf{X} \geq \mathbf{0}$$

Constrained colaborative sparse regression (CCSR)

$$\min_{\mathbf{X}} \ (1/2)\|\mathbf{AX} - \mathbf{Y}\|_F^2 + \lambda\|\mathbf{X}\|_{2,1}$$
$$\text{subject to: } \ \mathbf{X} \geq \mathbf{0}, \ \ \mathbf{1}_n^T\mathbf{X} = \mathbf{1}_N^T$$

$$\|\mathbf{X}\|_{2,1} := \textstyle\sum_{i=1}^{n} \|\mathbf{x}^i\|_2$$

Constrained total variation (CTV)

$$\min_{\mathbf{X}} \ (1/2)\|\mathbf{AX} - \mathbf{Y}\|_F^2 + \lambda_1\|\mathbf{X}\|_1 + \lambda_2\phi_{TV}(\mathbf{X})$$
$$\text{subject to: } \ \mathbf{X} \geq \mathbf{0}$$

Convex optimization problems in SLU

Structure of the optimization problems: $\min_{\mathbf{X} \in C} \sum_{i=1}^{J} g_i(\mathbf{X})$

convex functions (g_i)

convex set (C)

$$\Leftrightarrow \min_{\mathbf{X}} \sum_{i=1}^{J} g_i(\mathbf{X}) + \iota_C(\mathbf{X})$$

Source of difficulties: large scale ($n \times N \gtrsim 10^7$); nonsmoothness

Line of attack: alternating direction method of multiplies (ADMM)

[Glowinski, Marrocco, 75], [Gabay, Mercier, 76]

Alternating Direction Method of Multipliers (ADMM)

Unconstrained (convex) optimization problem: $\min_{\mathbf{z}\in\mathbb{R}^d} \; f_1(\mathbf{z}) + f_2(\mathbf{G}\,\mathbf{z})$

ADMM [Glowinski, Marrocco, 75], [Gabay, Mercier, 76]

$$\mathbf{z}_{k+1} = \arg\min_{\mathbf{z}} f_1(\mathbf{z}) + \frac{\mu}{2}\|\mathbf{G}\,\mathbf{z} - \mathbf{u}_k - \mathbf{d}_k\|^2$$

$$\mathbf{u}_{k+1} = \arg\min_{\mathbf{u}} f_2(\mathbf{u}) + \frac{\mu}{2}\|\mathbf{G}\,\mathbf{z}_{k+1} - \mathbf{u} - \mathbf{d}_k\|^2$$

$$\mathbf{d}_{k+1} = \mathbf{d}_k - (\mathbf{G}\,\mathbf{z}_{k+1} - \mathbf{u}_{k+1})$$

Interpretations: variable splitting + augmented Lagrangian + NLBGS;

Douglas-Rachford splitting on the dual [Eckstein, Bertsekas, 92];

split-Bregman approach [Goldstein, Osher, 08]

A Cornerstone Result on ADMM [Eckstein, Bertsekas, 1992]

Consider the problem

$$\min_{\mathbf{z}\in\mathbb{R}^d} \; f_1(\mathbf{z}) + f_2(\mathbf{G}\,\mathbf{z})$$

Let f_1 and f_2 be closed, proper, and convex and $\mathbf{G}$ have full column rank.

Let $(\mathbf{z}_k,\; k=0,1,2,...)$ be the sequence produced by ADMM, with $\mu > 0$, then, if the problem has a solution, say $\bar{\mathbf{z}}$, then

$$\lim_{k\rightarrow\infty} \mathbf{z}_k = \bar{\mathbf{z}}$$

The theorem also allows for inexact minimizations, as long as the errors are absolutely summable.

Convergence rate: $O(1/\varepsilon)$ [He, Yuan, 2011]

ADMM for two or more functions [Figueiredo, B-D, 09]

Consider a more general problem $$\min_{\mathbf{z}\in\mathbb{R}^d} \sum_{j=1}^{J} g_j(\mathbf{H}^{(j)}\mathbf{z}) \qquad (P)$$

$g_j : \mathbb{R}^{p_j} \to \bar{\mathbb{R}}$ — Proper, closed, convex functions

$\mathbf{H}^{(j)} \in \mathbb{R}^{p_j \times d}$ — Arbitrary matrices

There are many ways to write (P) as $\min_{\mathbf{z}\in\mathbb{R}^d} f_1(\mathbf{z}) + f_2(\mathbf{G}\,\mathbf{z})$

We adopt:

$$f_1(\mathbf{z}) = 0, \qquad \mathbf{G} = \begin{bmatrix} \mathbf{H}^{(1)} \\ \vdots \\ \mathbf{H}^{(J)} \end{bmatrix}, \qquad f_2\left(\begin{bmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(J)} \end{bmatrix}\right) = \sum_{j=1}^{J} g_j(\mathbf{u}^{(j)})$$

Another approach in [Goldfarb, Ma, 09, 11]

Applying ADMM to more than two functions

$$\mathbf{z}_{k+1} = \left[\sum_{j=1}^{J} (\mathbf{H}^{(j)})^* \mathbf{H}^{(j)}\right]^{-1} \left(\sum_{j=1}^{J} (\mathbf{H}^{(j)})^* \left(\mathbf{u}_k^{(j)} + \mathbf{d}_k^{(j)}\right)\right)$$

$$\mathbf{u}_{k+1}^{(1)} = \arg\min_{\mathbf{u}} g_1(\mathbf{u}) + \frac{\mu}{2}\|\mathbf{u} - \mathbf{H}^{(1)}\mathbf{z}_{k+1} + \mathbf{d}_k^{(1)}\|^2 = \text{prox}_{g_1/\mu}(\mathbf{H}^{(1)}\mathbf{z}_{k+1} - \mathbf{d}_k^{(j)})$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\mathbf{u}_{k+1}^{(J)} = \arg\min_{\mathbf{u}} g_J(\mathbf{u}) + \frac{\mu}{2}\|\mathbf{u} - \mathbf{H}^{(J)}\mathbf{z}_{k+1} + \mathbf{d}_k^{(J)}\|^2 = \text{prox}_{g_1/\mu}(\mathbf{H}^{(1)}\mathbf{z}_{k+1} - \mathbf{d}_k^{(j)})$$

$$\mathbf{d}_{k+1}^{(1)} = \mathbf{d}_k^{(1)} - (\mathbf{H}^{(1)}\mathbf{z}_{k+1} - \mathbf{u}_{k+1}^{(1)})$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\mathbf{d}_{k+1}^{(J)} = \mathbf{d}_k^{(J)} - (\mathbf{H}^{(J)}\mathbf{z}_{k+1} - \mathbf{u}_{k+1}^{(J)})$$

Conditions for easy applicability: inexpensive proximity operators

inexpensive matrix inversion

Constrained sparse regression (CSR)

Problem $\min_{\mathbf{X}} (1/2)\|\mathbf{AX}-\mathbf{Y}\|_F^2 + \lambda\|\mathbf{X}\|_1, \quad \text{subject to: } \mathbf{X} \geq \mathbf{0}$

Equivalent formulation $\min_{\mathbf{X}} (1/2)\|\mathbf{AX}-\mathbf{Y}\|_F^2 + \lambda\|\mathbf{X}\|_1 + \iota_{\mathbb{R}_+}(\mathbf{X})$

indicator of the first orthant

Template: $\min_{\mathbf{X}} \sum_{j=1}^{J=3} g_j(\mathbf{H}^{(j)}\mathbf{X})$

Mapping:

$$\mathbf{H}^{(1)} = \mathbf{A}, \quad g_1(\mathbf{Z}) = (1/2)\|\mathbf{Z}-\mathbf{Y}\|_F^2$$

$$\mathbf{H}^{(2)} = \mathbf{I}, \quad g_2(\mathbf{Z}) = \lambda\|\mathbf{Z}\|_1$$

$$\mathbf{H}^{(3)} = \mathbf{I}, \quad g_3(\mathbf{Z}) = \iota_{\mathbb{R}_+}(\mathbf{Z})$$

Proximity operators:

$$\text{prox}_{g_1/\mu}(\mathbf{W}) = \frac{\mathbf{Y}+\mu\mathbf{W}}{1+\mu}$$

$$\text{prox}_{g_2/\mu}(\mathbf{W}) = \text{soft}(\mathbf{W}, \lambda/\mu)$$

$$\text{prox}_{g_3/\mu}(\mathbf{W}) = \max\{\mathbf{0}, \mathbf{W}\}$$

Matrix inversion can be precomputed

(typical sizes 200~300 x 500~1000)

$$\left[\sum_{j=1}^{J}(\mathbf{H}^{(j)})^*\,\mathbf{H}^{(j)}\right]^{-1} = \left[\mathbf{A}^T\mathbf{A}+2\mathbf{I}\right]^{-1}$$

Spectral unmixing by split augmented Lagrangian (SUnSAL) [B-D, Figueiredo, 2010]

Related algorithm (split-Bregman view) in [Szlam, Guo, Osher, 2010]

Concluding remarks

- Sparse regression framework, used with care, yields effective hyperspectral unmixing results.

- Critical factors
  - High mutual coherence of the hyperspectral libraries
  - Non-linear mixing and noise
  - Acquisition and calibration of hyperspectral libraries

- Favorable factors
  - Hyperspectral mixtures are highly sparse

- ADMM is a very flexible and efficient tool solving the hyperspectral sparse regression optimization problems

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