WAVES AND EDDIES IN ROTATING TURBULENCE

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THE NAVIER–STOKES EQUATIONS

- Momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla P + \nu\nabla^2\mathbf{v} + \mathbf{F} \qquad \nabla\cdot\mathbf{v} = 0$$

- P is the pressure, F an external force, ν the kinematic viscosity, and **v** the velocity; incompressibility is assumed.
- Quadratic invariants (**F** = 0, ν =0):

$$E = \int \mathbf{v}^2 \, d^3x$$

$$H = \int \mathbf{v}\cdot\boldsymbol{\omega} \, d^3x \qquad \boldsymbol{\omega} = \nabla\times\mathbf{v}$$

- Reynolds numbers:

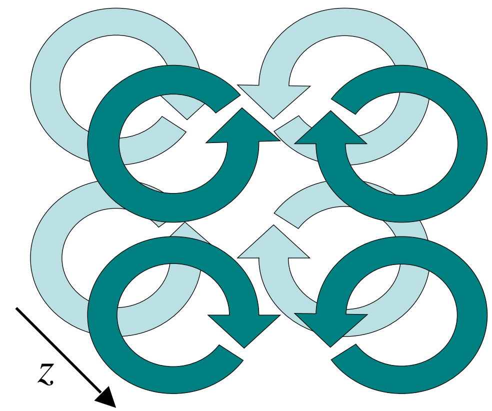
$$Re = UL / \nu \qquad R_\lambda = U\lambda / \nu$$

where L is the integral scale and λ the Taylor scale.

THE ENERGY CASCADE

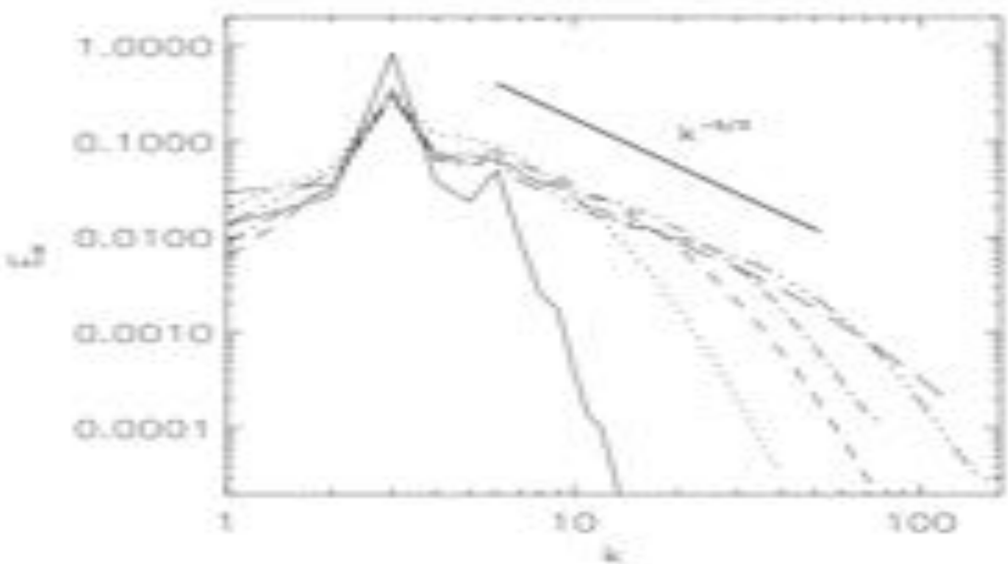
Starting from

$$\mathbf{v} = \begin{bmatrix} \sin(k_0 x)\cos(k_0 y)\cos(k_0 z) \\ -\cos(k_0 x)\sin(k_0 y)\cos(k_0 z) \\ 0 \end{bmatrix}$$



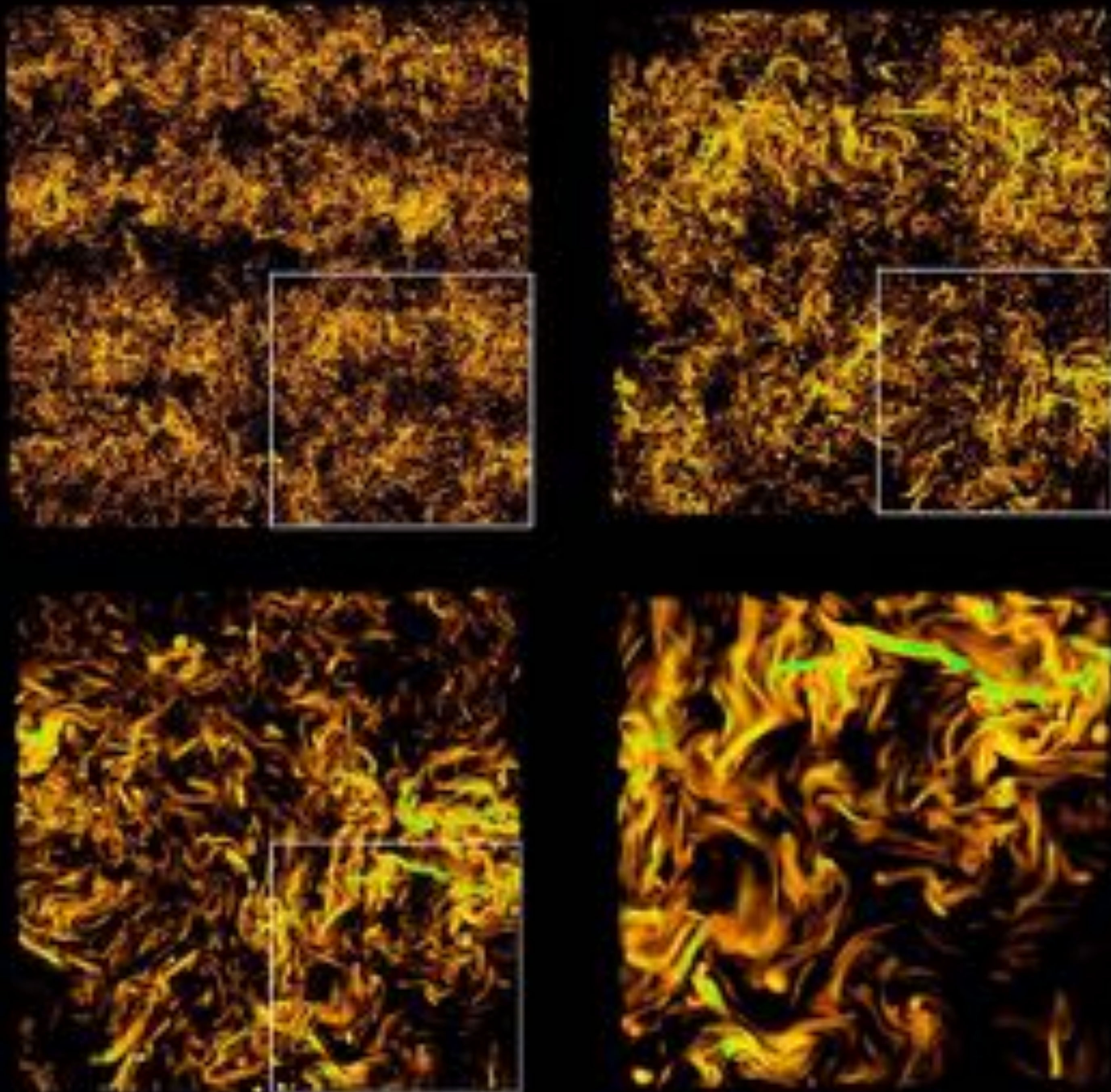
as initial condition, and replacing in the Navier-Stokes equation

$$\frac{\partial v_x}{\partial t} = \frac{k_0 \sin(2k_0 x)}{8}\left[\cos(2k_0 z) - \cos(2k_0 y)\right] - 3k_0^2 \nu \cos(k_0 x)\sin(k_0 y)\sin(k_0 z)$$



- This process can be repeated, and smaller eddies are created until reaching the scale where the dissipative term dominates! Taylor & Green, Proc. Roy. Soc. A 151, 421 (1935).

2048^3



TURBULENCE: THE NAVIER-STOKES EQUATIONS

- This leads naturally to a Fourier representation for the velocity in the momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{v} + \mathbf{F} \qquad \nabla \cdot \mathbf{v} = 0$$

- Fourier representation

$$\mathbf{v}(\mathbf{x}, t) = \int d^3k \; e^{i\mathbf{k}\cdot\mathbf{x}} \; \tilde{\mathbf{v}}(\mathbf{k}, t)$$

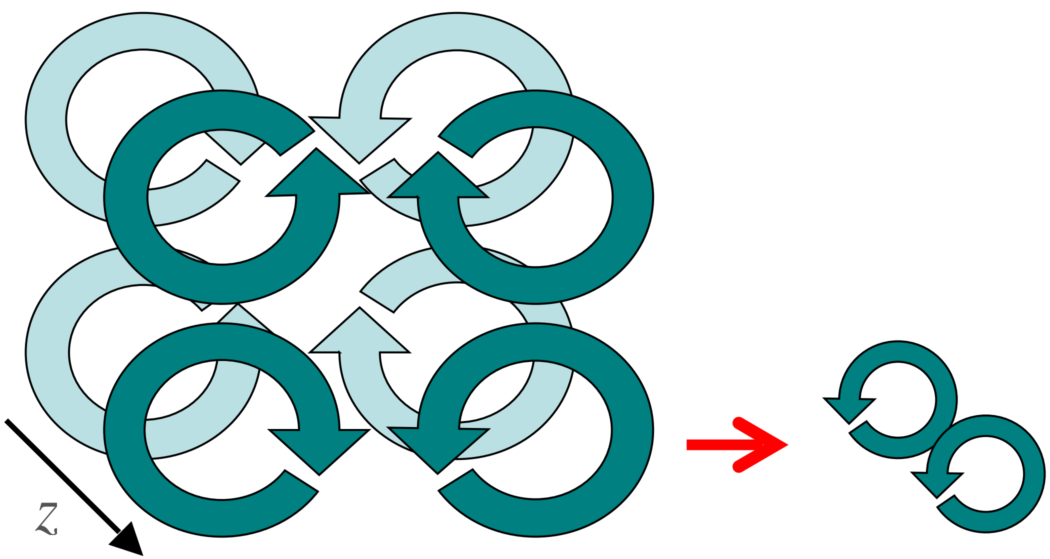
- Energy spectrum $S(\mathbf{k}) \sim \langle |\mathbf{v}(\mathbf{k})|^2 \rangle$

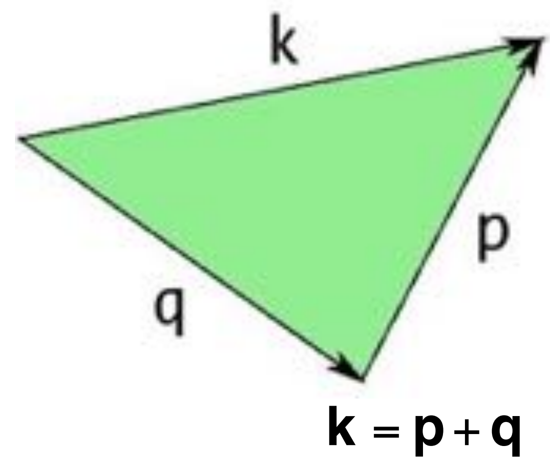
- Large, energy containing eddies with correlation scale L. Small scale fluctuations with wavenumber $k >> 1/L$.

ENERGY TRANSFER AND TRIADIC INTERACTIONS

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{v} + \mathbf{F}$$

$$\Rightarrow \frac{\partial \mathbf{v}_k}{\partial t} = -\int_{p,q} \left[\left(\mathbf{v}_p \cdot \nabla \right) \mathbf{v}_q \right] dp\,dq - i\mathbf{k}P_k - \nu k^2 \mathbf{v}_k + \mathbf{F}_k$$





ROTATING FLOWS

- Momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla \mathcal{P} + \nu \nabla^2 \mathbf{u} + \mathbf{F} \qquad \nabla \cdot \mathbf{u} = 0$$

$\mathcal{P}$ is the pressure, $\mathbf{F}$ an external force, $\boldsymbol{\nu}$ the kinematic viscosity, $\boldsymbol{\Omega}$ the angular velocity, and $\mathbf{u}$ the velocity; incompressibility is assumed.

- Quadratic invariants ($\mathbf{F} = 0, \nu = 0$):

$$E = \int \mathbf{u}^2 \, \mathrm{d}^3x$$

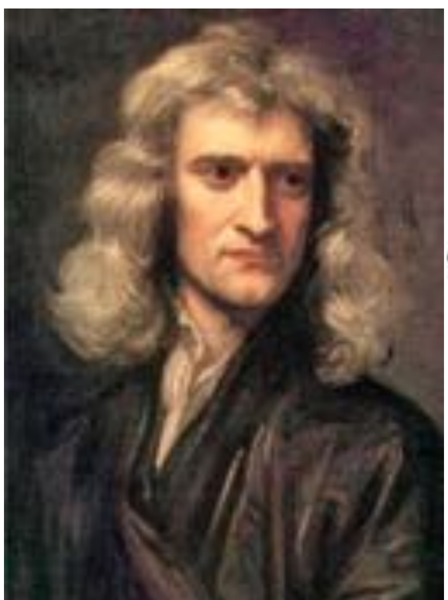
$$H = \int \mathbf{u} \cdot \omega \, \mathrm{d}^3x \qquad \omega = \nabla \times \mathbf{u}$$

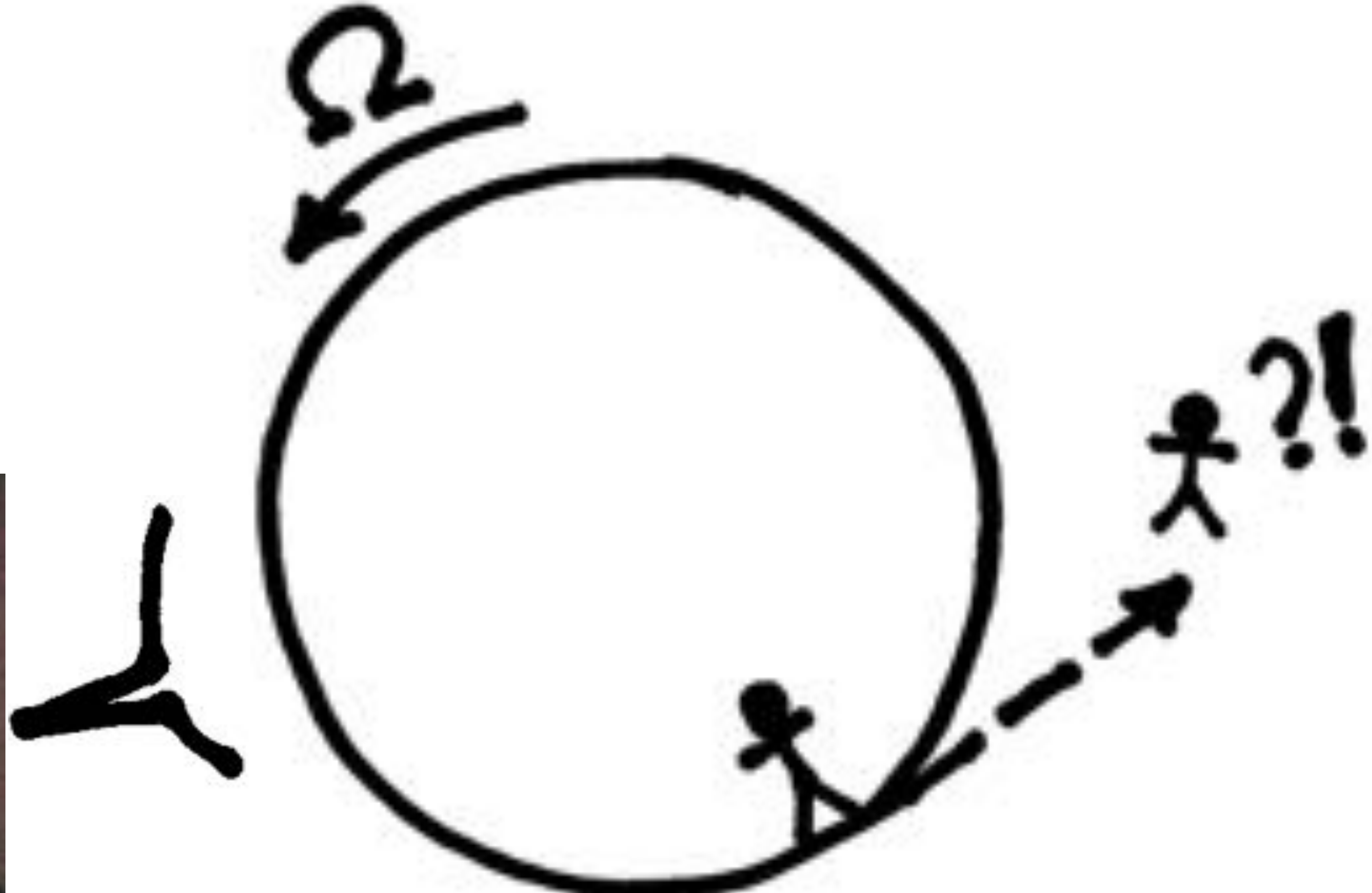
- Reynolds, Rossby, and Ekman numbers

$$Re = \frac{L_F U}{\nu} \qquad Ro = \frac{U}{2\Omega L_F} \qquad Ek = \frac{Ro}{Re} = \frac{\nu}{2\Omega L_F^2}$$

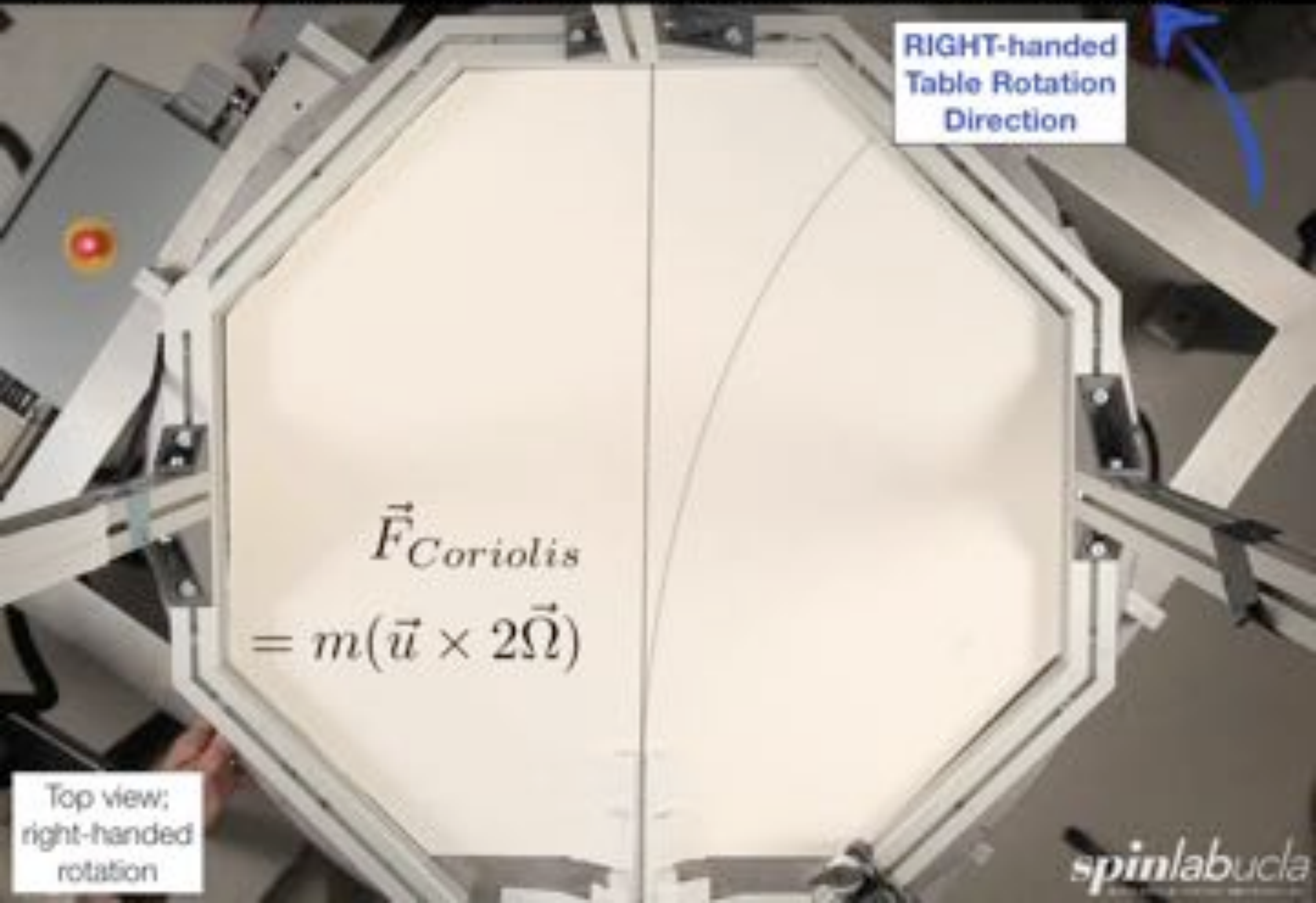
where L_F is the forcing scale.

INERTIA





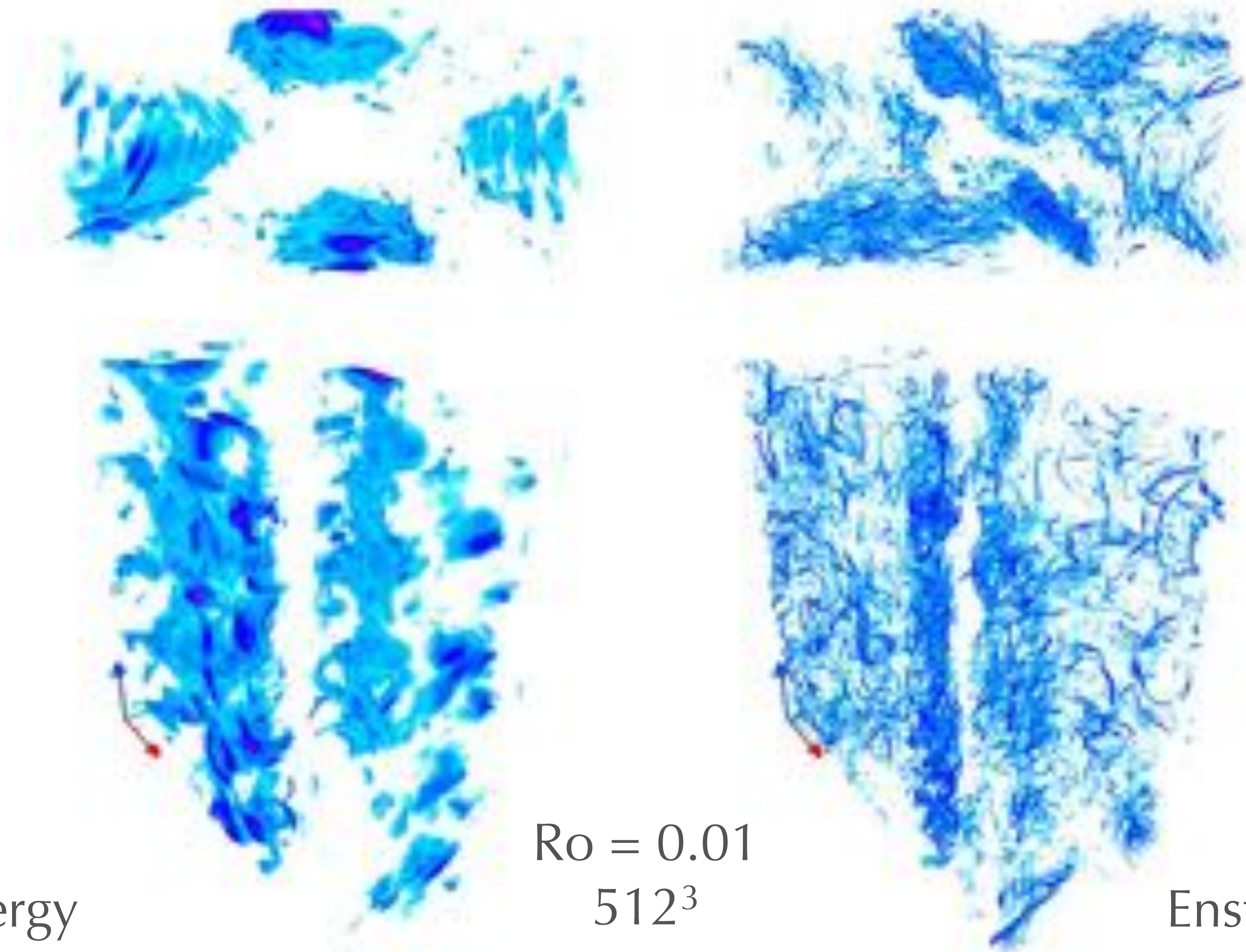
Coriolis Deflection: Effect of Rotation Direction



NUMERICAL SIMULATIONS

- GHOST code, publicly available.
- Visualizations done with VAPOR, publicly available.
- Periodic boundary conditions.
- Bounded domain.
- Discrete set of inertial waves.
- The number of modes that satisfy resonance conditions depends on wavenumber.
- Natural representation in terms of Fourier modes.
- External forces are body forces.

NON-HELICAL ROTATING TURBULENCE



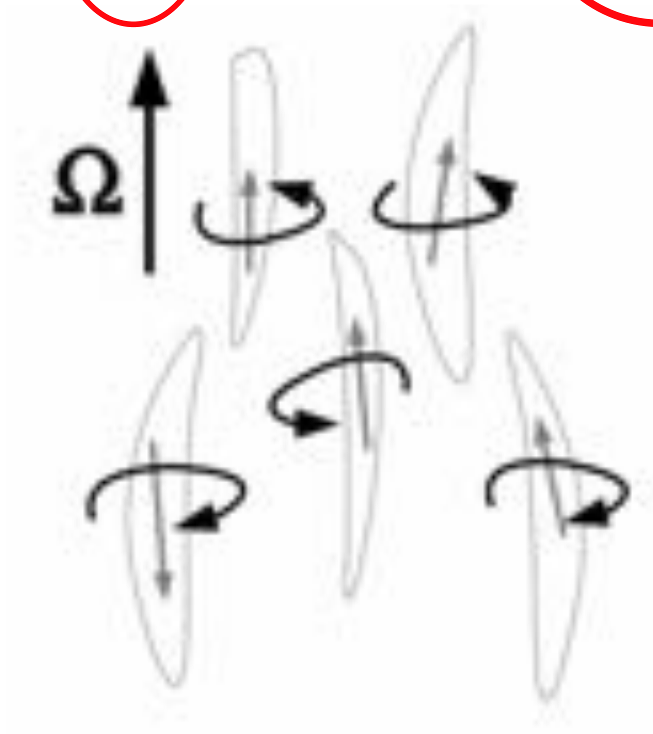
Energy

Ro = 0.01
512^3

Enstrophy

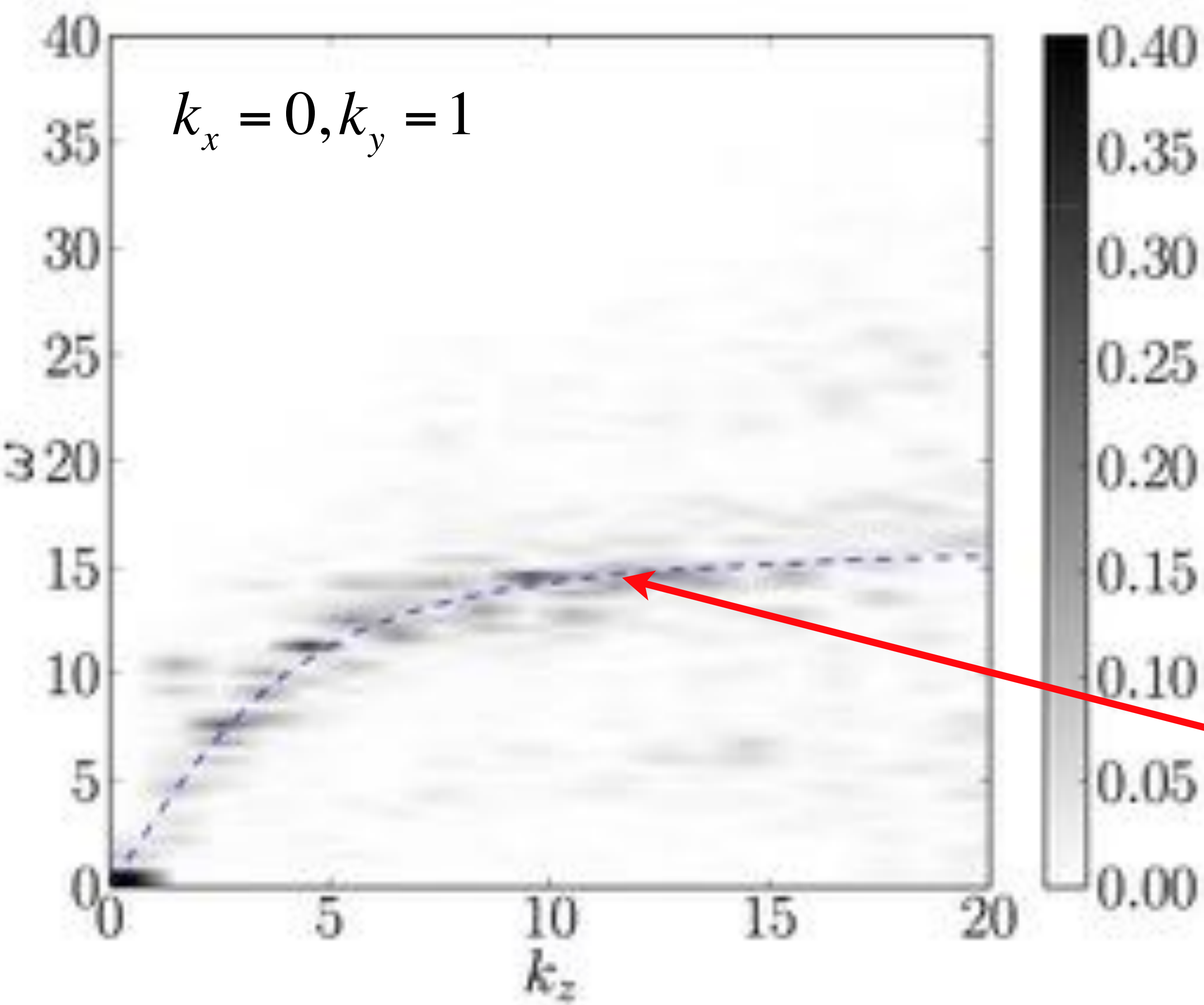
WAVES IN ROTATING FLOWS

$$\frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} + 2\Omega \times \mathbf{u} = -\nabla \mathcal{P} + \nu \nabla^2 \mathbf{u} + \mathbf{F}$$



$$\omega = \pm\Omega \frac{k_z}{k} \qquad u_x = \pm i u_y$$

WAVES OR EDDIES?



$$\frac{E(k,\omega)}{E(k)}$$

$$\omega = \Omega \frac{k_z}{k}$$

Clark di Leoni, Cobelli, Mininni, Dmitruk & Matthaeus, PoF 26, 035106 (2014).
See also Hopfinger et al 1982, Bewley et al 2007, Bordes, Moisy, Dauxois and Cortet 2012

WAVES IN ROTATING FLOWS

$$\omega = \pm\Omega \frac{k_z}{k} \qquad u_x = \pm i u_y$$

This leads to a natural decomposition in spectral space:

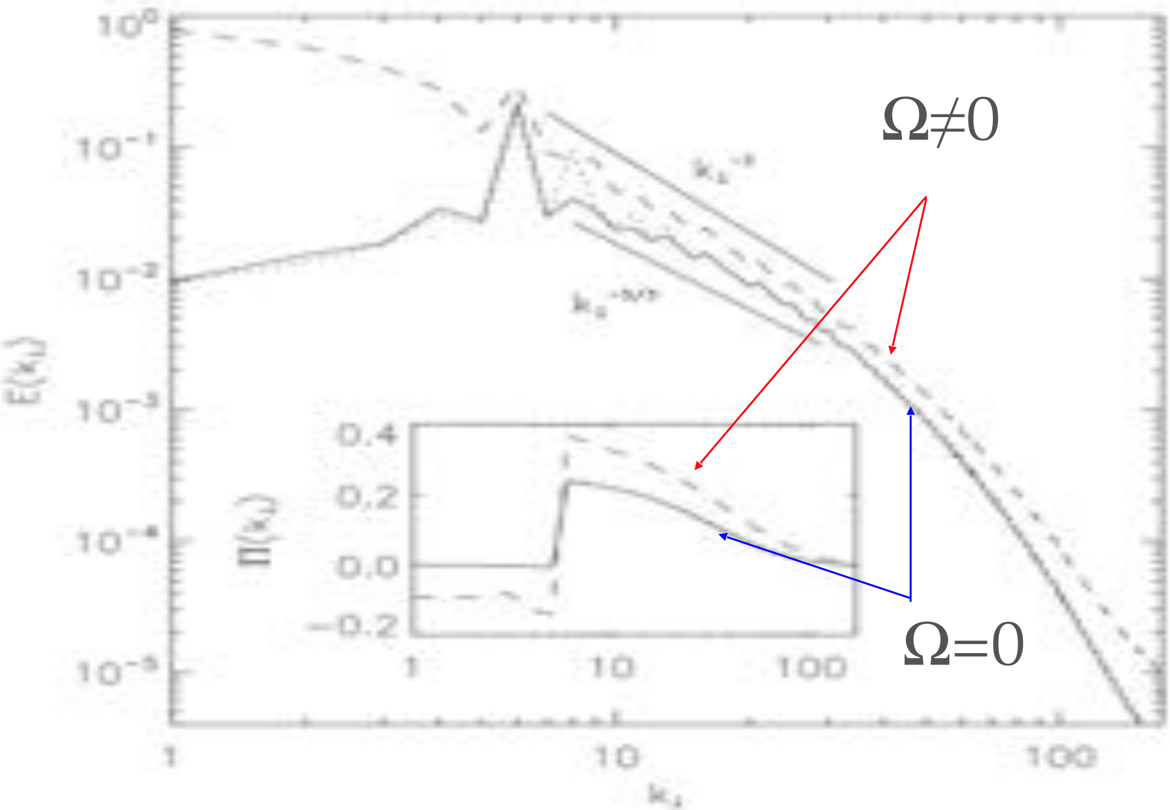
- 3D modes are "waves" (or "fast" modes, for sufficiently large Ro).
- 2D modes are "eddies" (or "slow" modes).

$$\mathbf{u}(\mathbf{k}) = \begin{cases} \mathbf{u}_{3D}(\mathbf{k}) & \text{if } \mathbf{k} \in W_k \\ \mathbf{u}_\perp(\mathbf{k}_\perp) + w(\mathbf{k}_\perp)\hat{z} & \text{if } \mathbf{k} \in V_k \end{cases}$$

$$W_k := \{\mathbf{k} \text{ s.t. } |\mathbf{k}| \neq 0 \text{ and } k_{||} \neq 0\}$$

$$V_k := \{\mathbf{k} \text{ s.t. } |\mathbf{k}| \neq 0 \text{ and } k_{||} = 0\}$$

ENERGY SPECTRUM OF ROTATING (NON-HELICAL) FLOWS



Non-helical case:

- An inverse cascade of energy develops for small Ro.
- The flow becomes anisotropic.
- The spectrum goes towards $k_{\perp}^{-2}$ as rotation is increased (Ro decreased).

Mininni, Alexakis & Pouquet, PoF **21**, 015108; Mininni & Pouquet, PRE **79**, 026304 (2009)

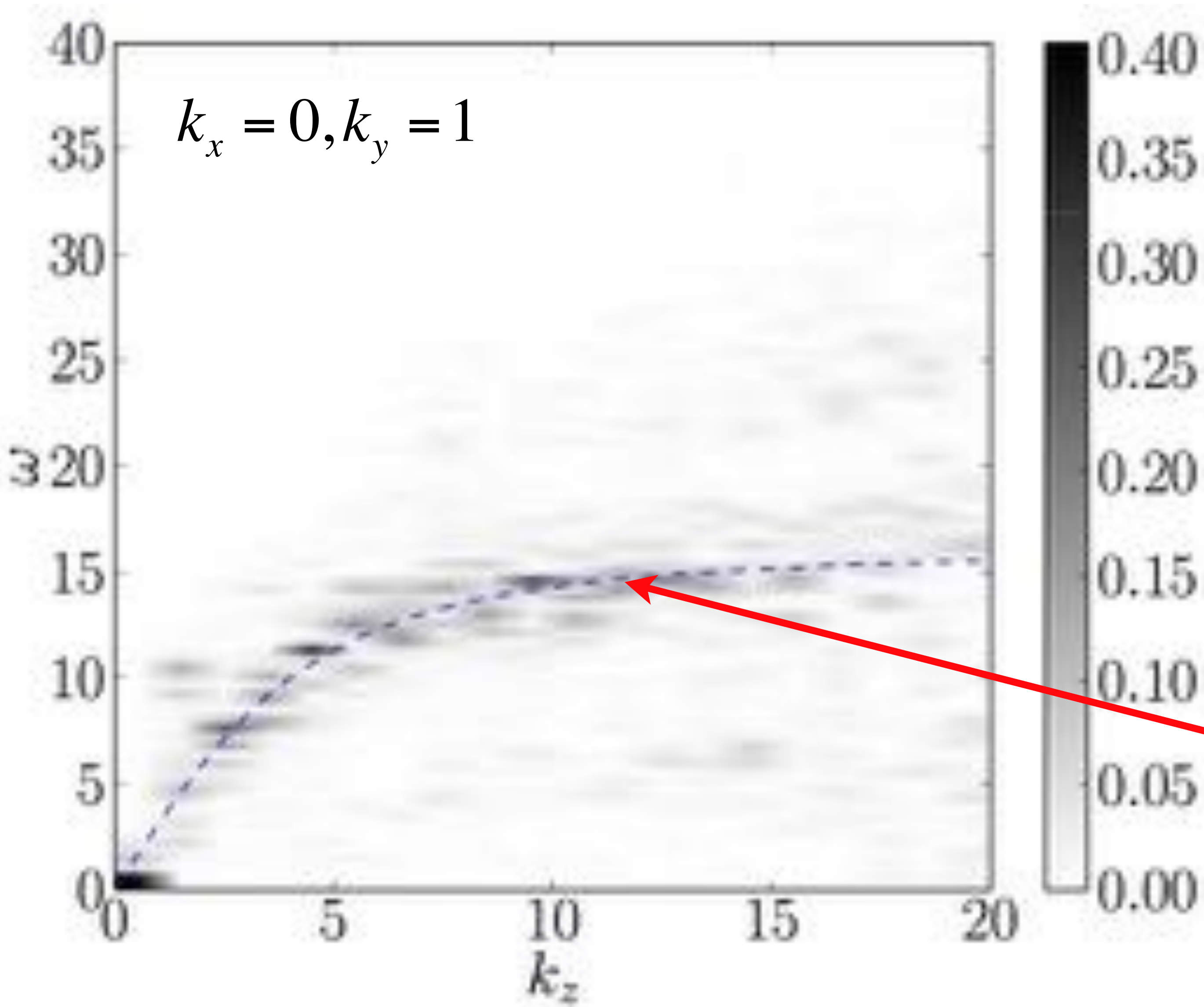
PHENOMENOLOGY OF ROTATING TURBULENCE

- The interaction of waves and eddies slows down the cascade (Cambon and Jacquin 1989, Cambon, Mansour, and Godeferd 1997).
- Following Kraichnan (1965) phenomenology, we can assume that the time to move energy across scales is increased by a factor τ_l / τ_Ω.
- The inverse of the transfer time then becomes $1/\tau_{NL} = \tau_\Omega / \tau_l^2$.
- As a result of the resonant interactions, the flow also becomes anisotropic, with $1/\tau_l \sim u_l / l_\perp$.
- The energy transferred between scales per unit of time is

$$\varepsilon \sim u_l^2/\tau_{NL} \sim u_l^4/l_\perp^2, \text{ and } u_l^2 \sim l_\perp.$$

- Then the energy spectrum is $E(k_\perp) \sim k_\perp^{-2}$ (Dubrulle 1992, Zhou 1995).
- A more detailed derivation using two-point closures can be found, e.g., in Cambon and Jacquin (1989).

SPATIO-TEMPORAL SPECTRUM



$$\frac{E(k,\omega)}{E(k)}$$

$$\omega = \Omega \frac{k_z}{k}$$

Clark di Leoni, Cobelli, Mininni, Dmitruk & Matthaeus, PoF 26, 035106 (2014).
See also Hopfinger et al 1982, Bewley et al 2007, Bordes, Moisy, Dauxois and Cortet 2012

DOMINANT DECORRELATION TIMES

Time scales:

- Wave period

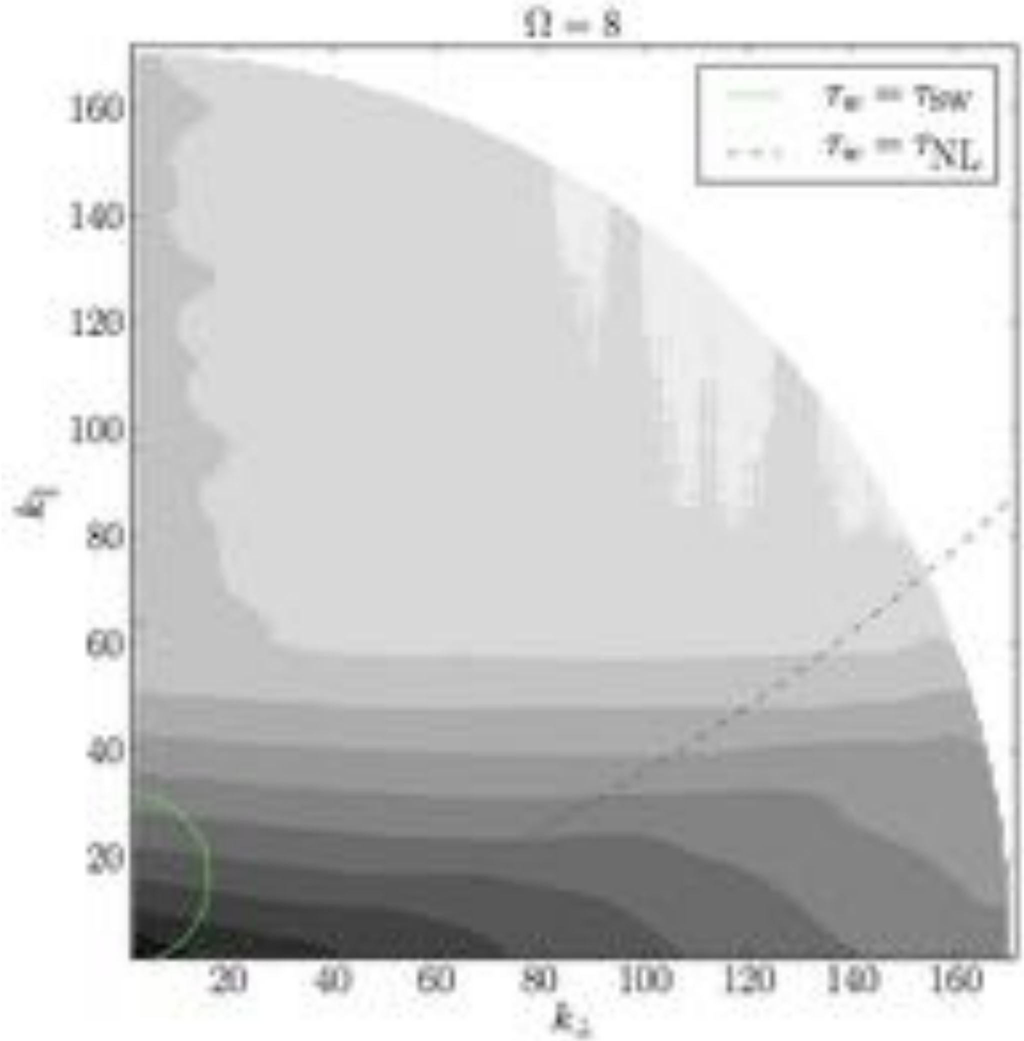
$$\tau_\omega(\mathbf{k}) = C_\omega \frac{k}{2\Omega k_{||}}$$

- Non-linear time

$$\tau_{\rm NL}(\mathbf{k}) = C_{\rm NL} \frac{1}{\epsilon^{1/4}\Omega^{1/4}k^{1/2}}$$

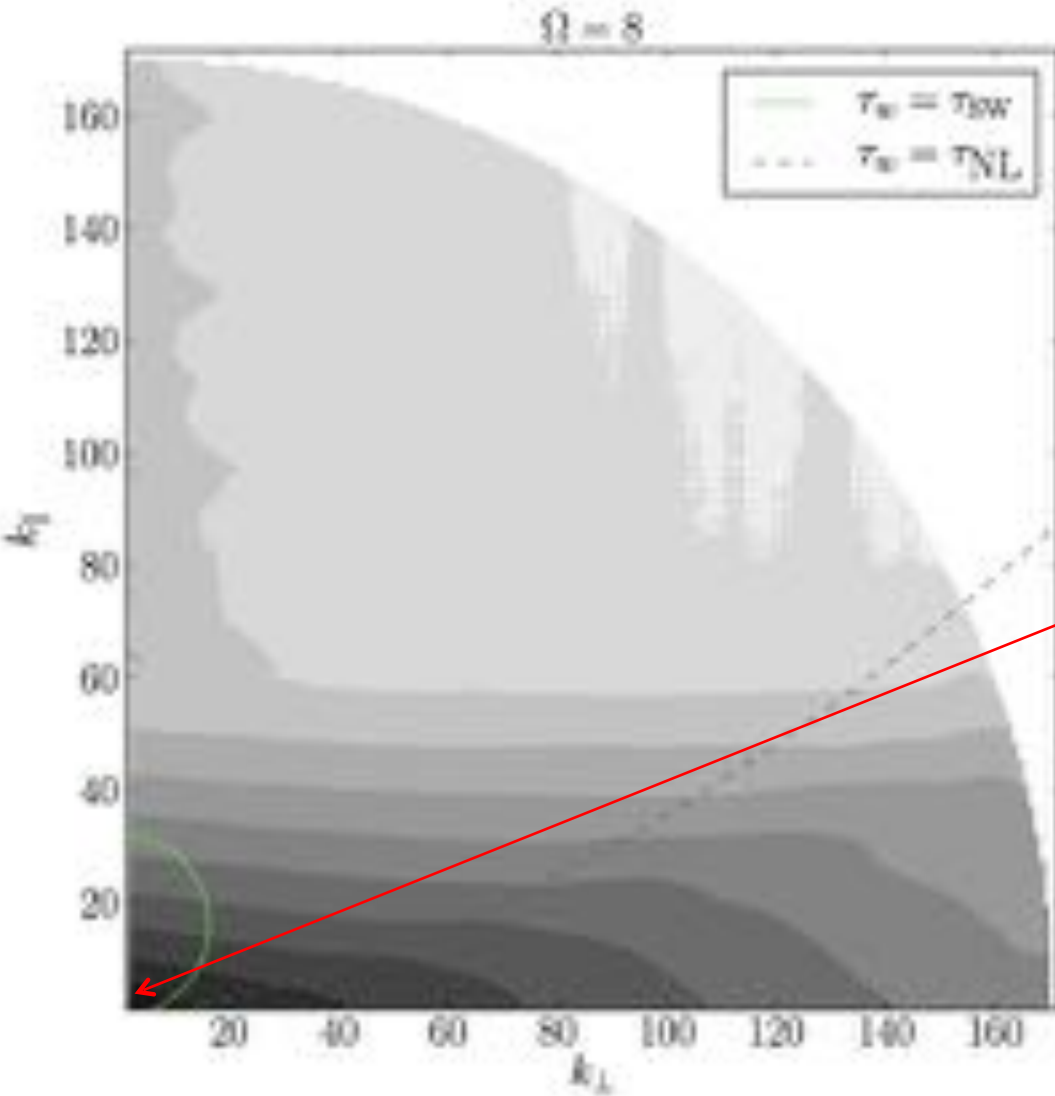
- Sweeping time

$$\tau_{\rm sw}(\mathbf{k}) = C_{\rm sw} \frac{1}{Uk}$$

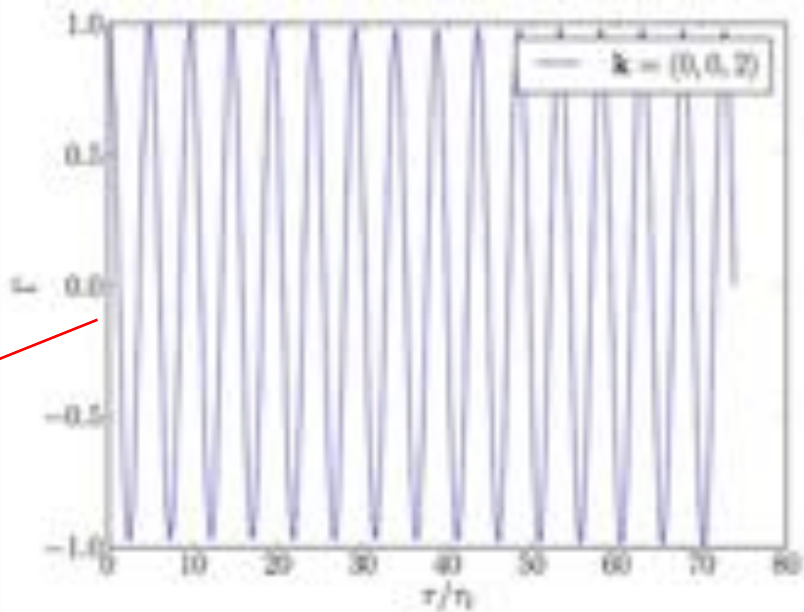


Clark di Leoni, Cobelli, Mininni, Dmitruk & Matthaeus, PoF 26, 035106 (2014)

DECORRELATION TIMES

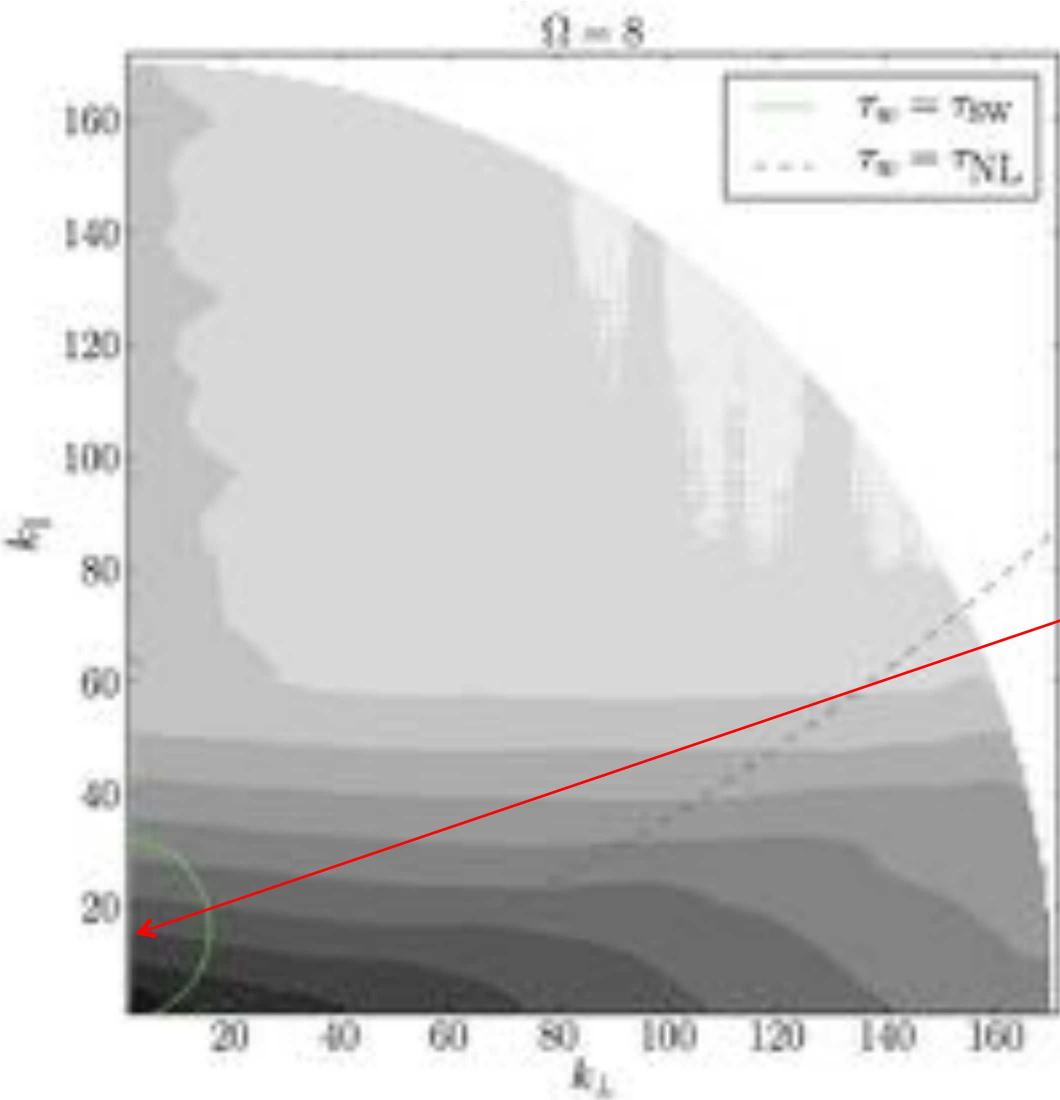


$$\Gamma_{ij}(\mathbf{k},\tau) = \frac{\langle \hat{u}_i^*(\mathbf{k},t)\hat{u}_j(\mathbf{k},t+\tau)\rangle_t}{\langle |\hat{u}_i^*(\mathbf{k},t)\hat{u}_j(\mathbf{k},t)|\rangle_t}$$



See also Fabier, Godeferd and Cambon (2010)

DECORRELATION TIMES



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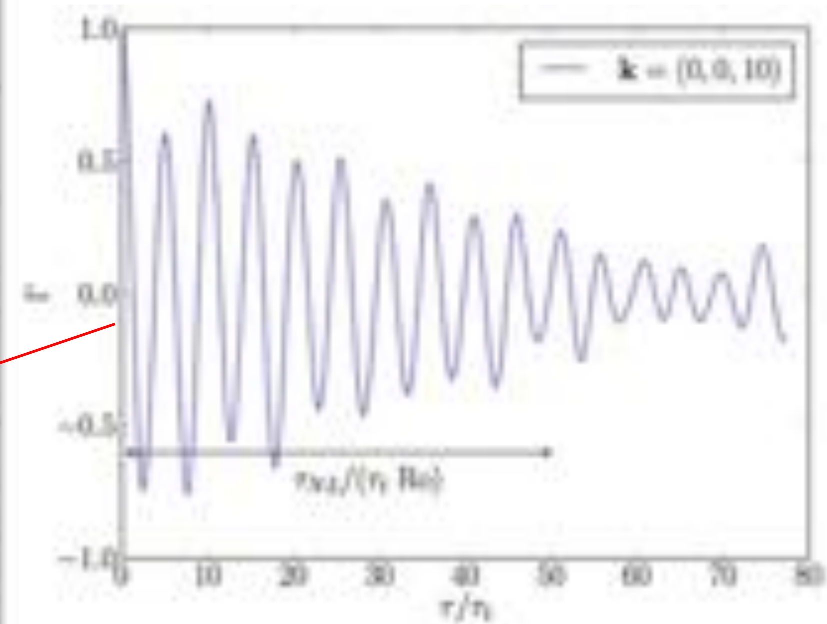
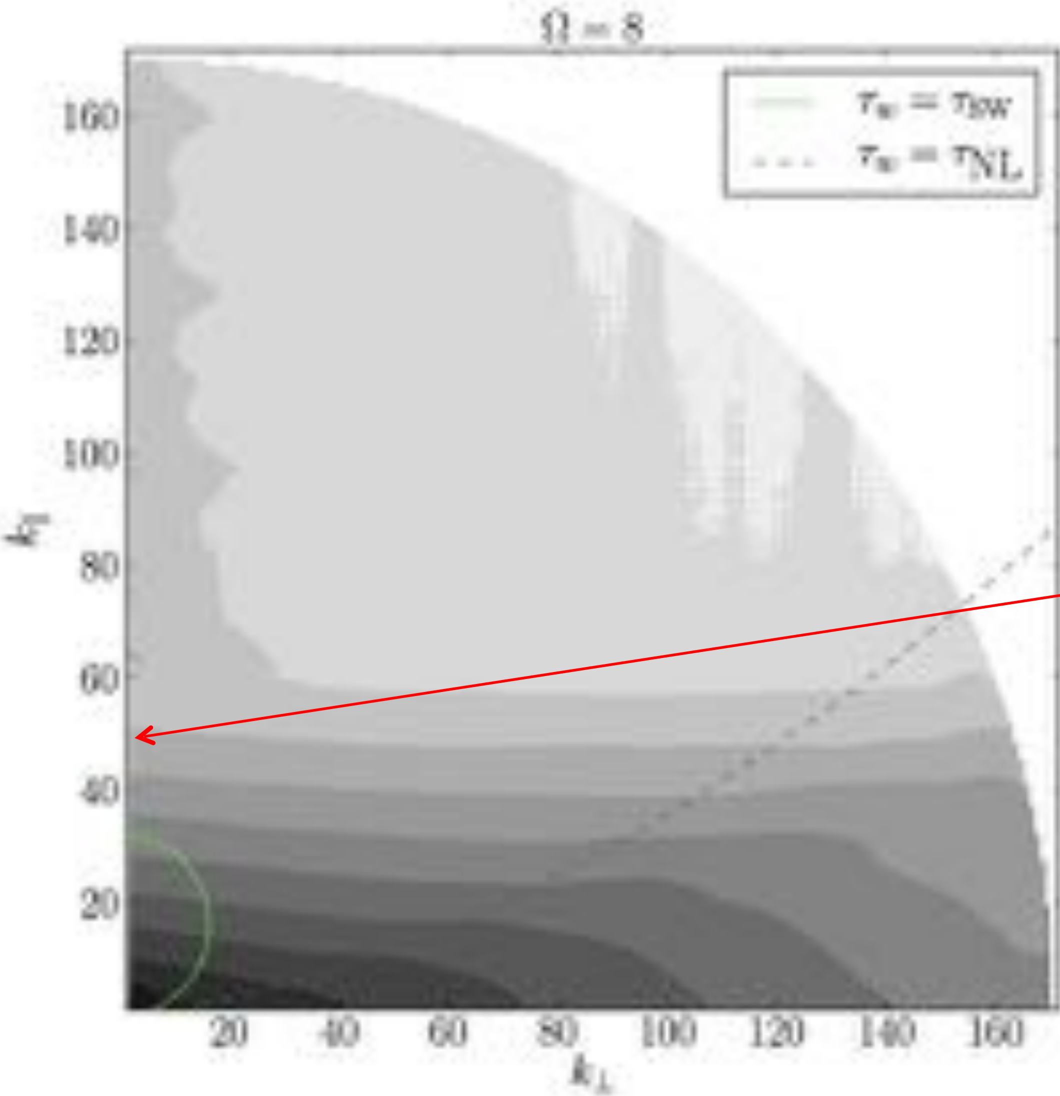


DECORRELATION TIMES



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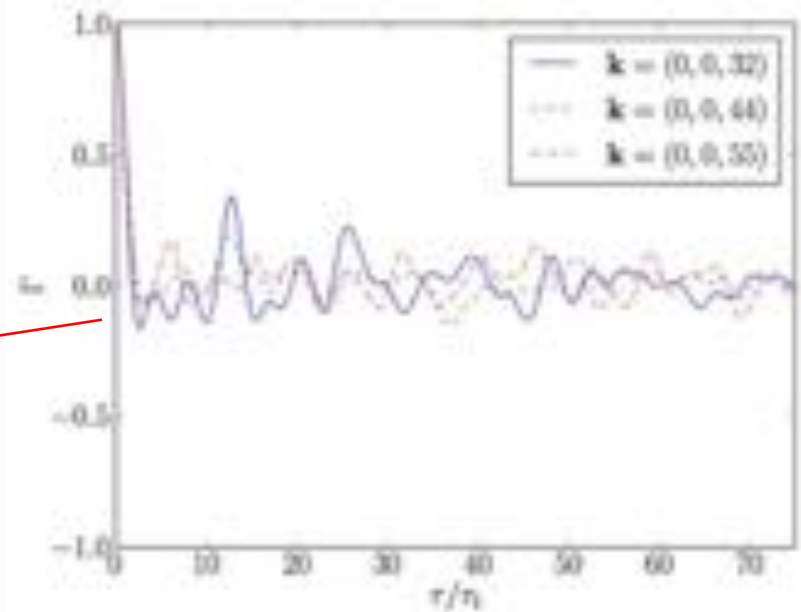
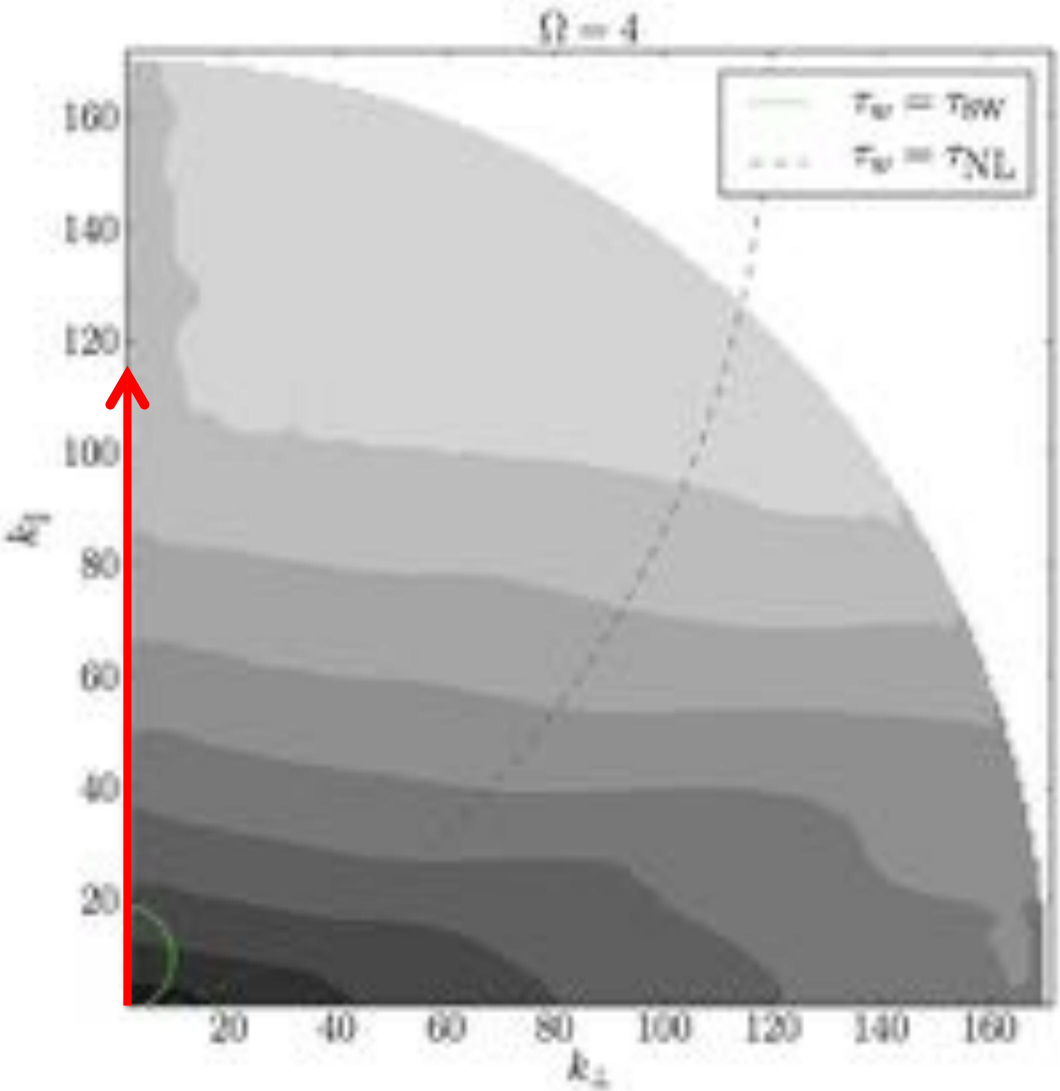
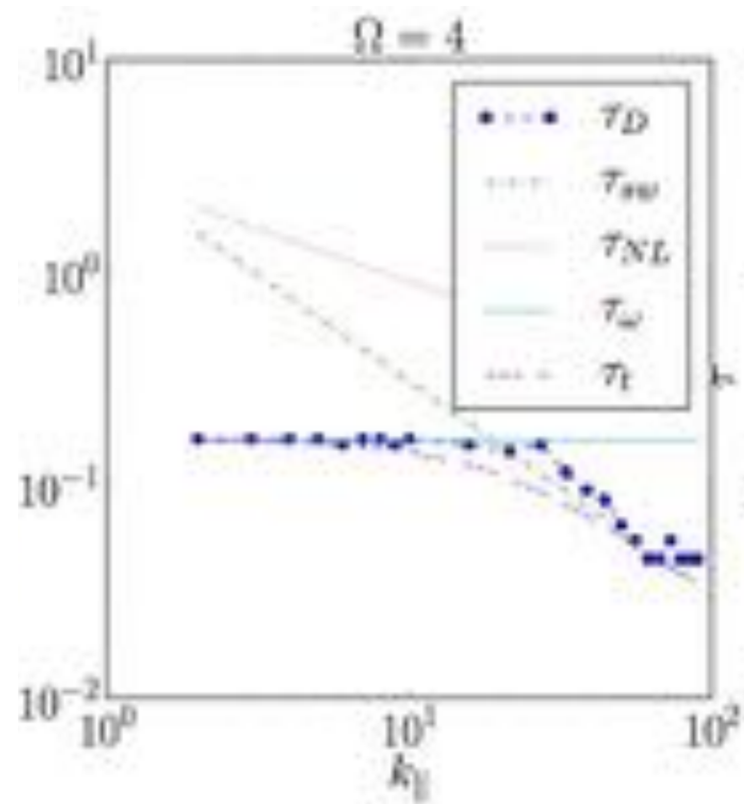


DECORRELATION TIMES

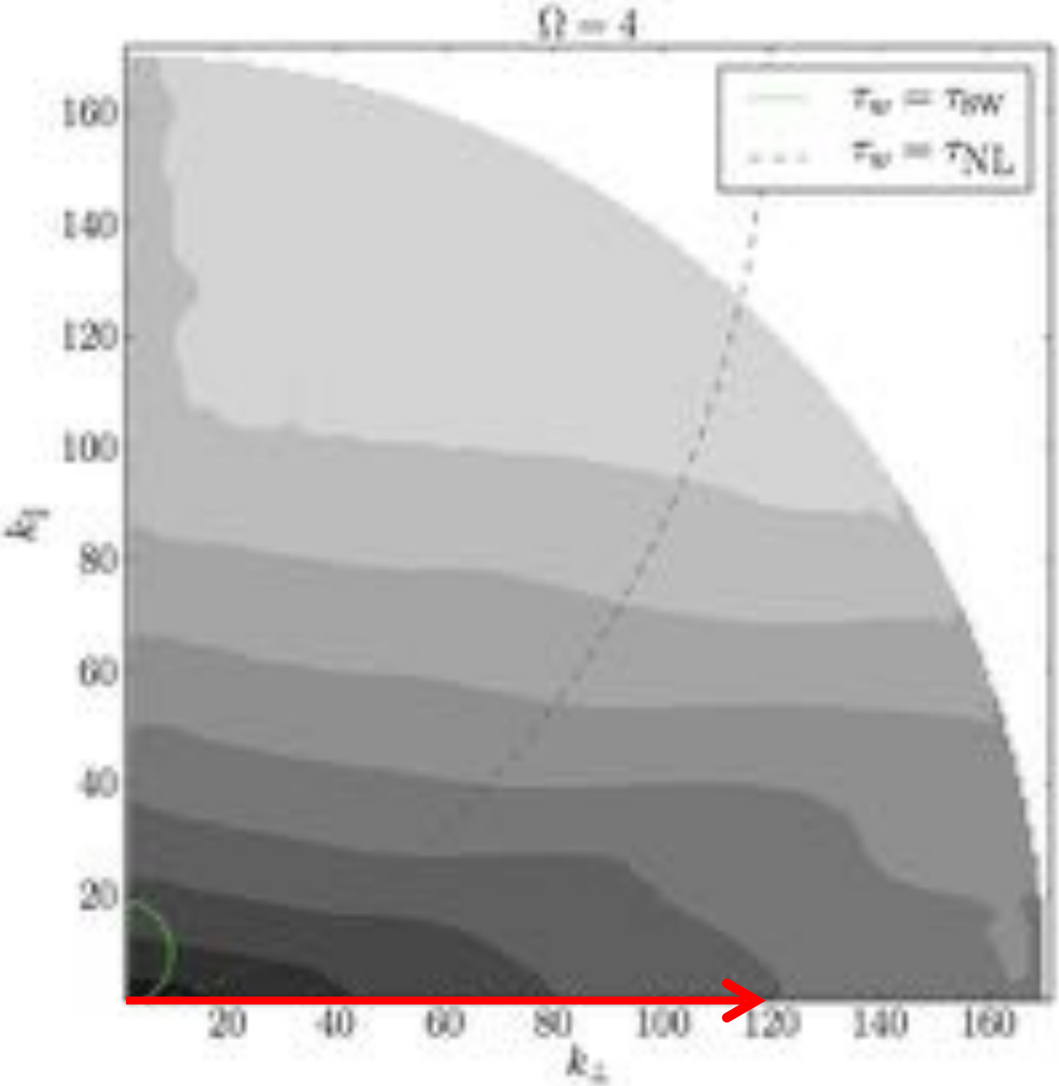


$$\left(\frac{1}{\tau_{\rm t}}\right)^2 = \left(\frac{1}{\tau_\omega}\right)^2 + \left(\frac{1}{\tau_{\rm sw}}\right)^2$$

See Cambon & Jacquin (1989)

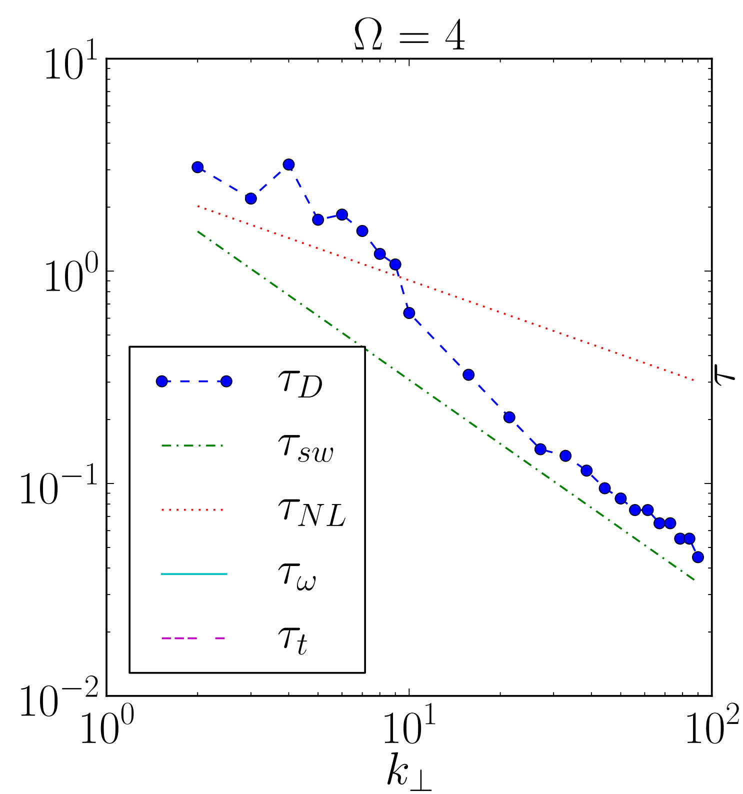


DECORRELATION TIMES

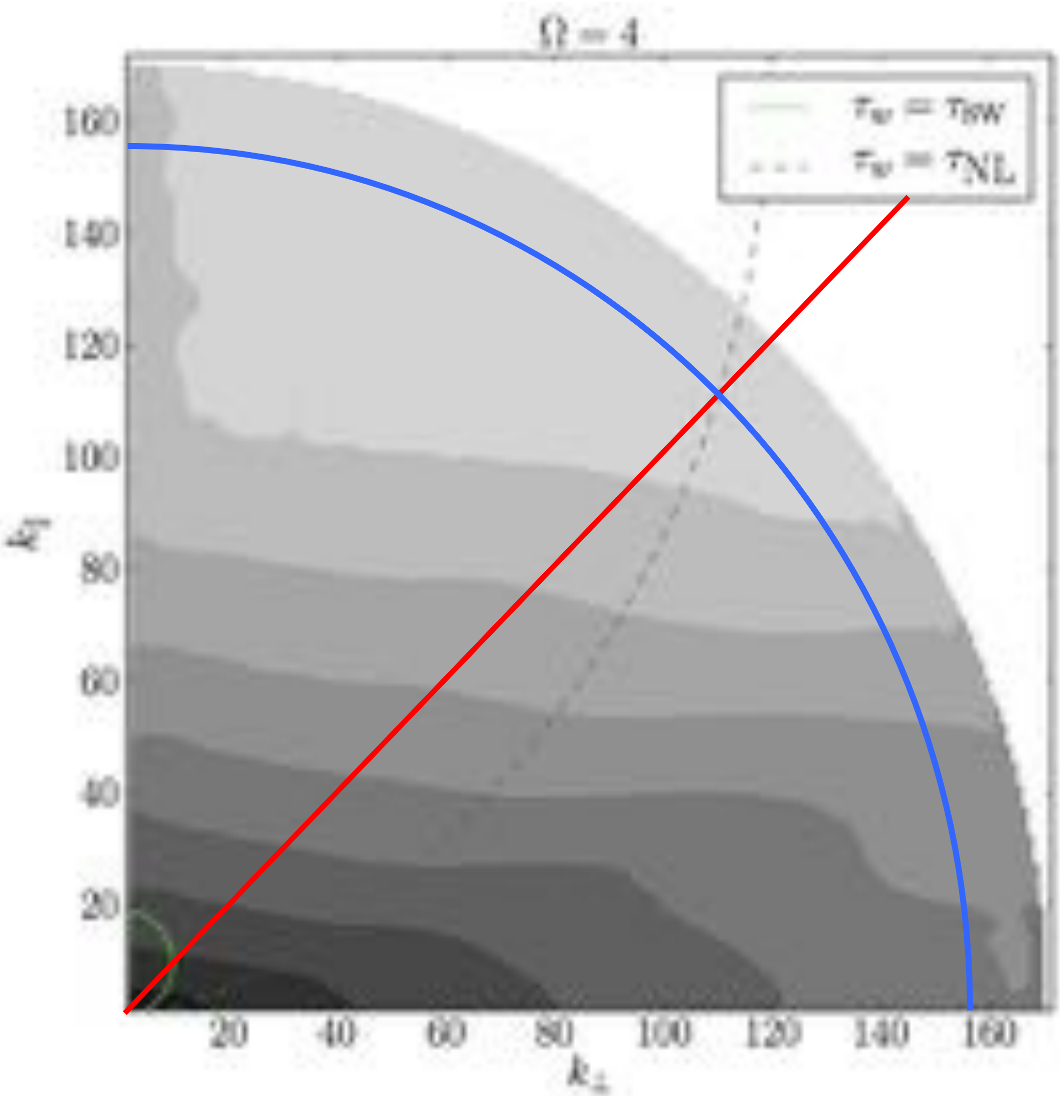


$$\left(\frac{1}{\tau_{\mathrm{t}}}\right)^2 = \left(\frac{1}{\tau_\omega}\right)^2 + \left(\frac{1}{\tau_{\mathrm{sw}}}\right)^2$$

See Cambon & Jacquin (1989)



RECOVERY OF ISOTROPY



If isotropy is recovered:

$$k_{\parallel} \approx k_{\perp}$$

$$\boldsymbol{\tau}_{\omega} \approx \boldsymbol{\tau}_{NL}$$

$$\frac{C_{\mathrm{NL}}}{\epsilon^{1/4}\Omega^{1/4}k_{\Omega}^{1/2}} = \frac{C_{\omega}}{\sqrt{2}\Omega}$$

$$\Rightarrow k_{\Omega} = C_{\Omega}\left(\frac{\Omega^{3}}{\epsilon}\right)^{1/2}$$

ISOTROPY VS. ANISOTROPY

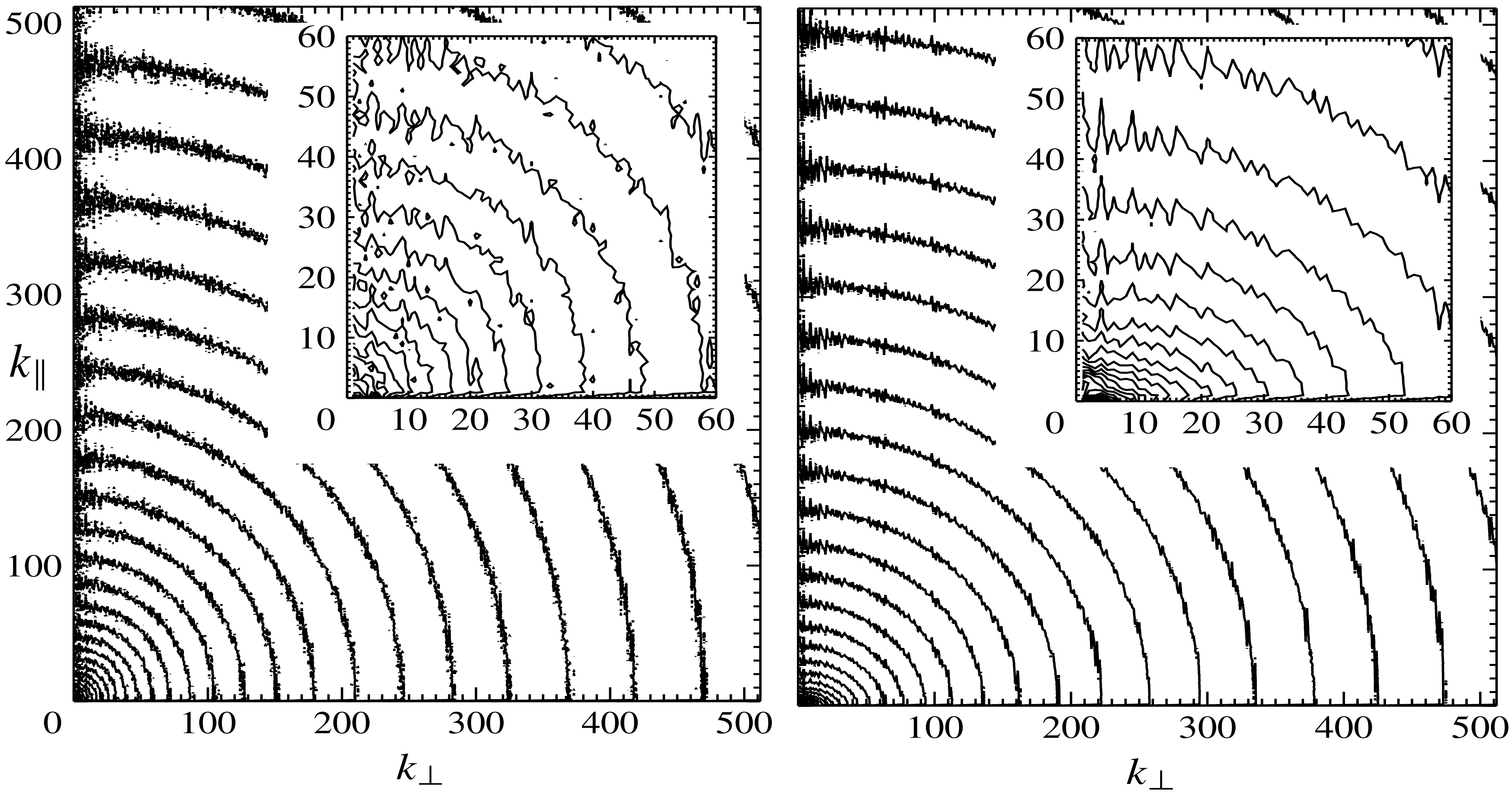
- Do rotating flows recover isotropy at small scales?
- Since we don't feel the rotation of the Earth, we know it should!
- How does rotating turbulence look like in that multi-scale case?
- We can expect the spectrum to be

$$E(k) = Ak^{-\alpha} + Bk^{-5/3}$$

(with $2 \leq \alpha \leq 2.5$). The transition between the two spectra should take place when the eddy turnover time becomes of the same order as the wave time (Zeman 1994):

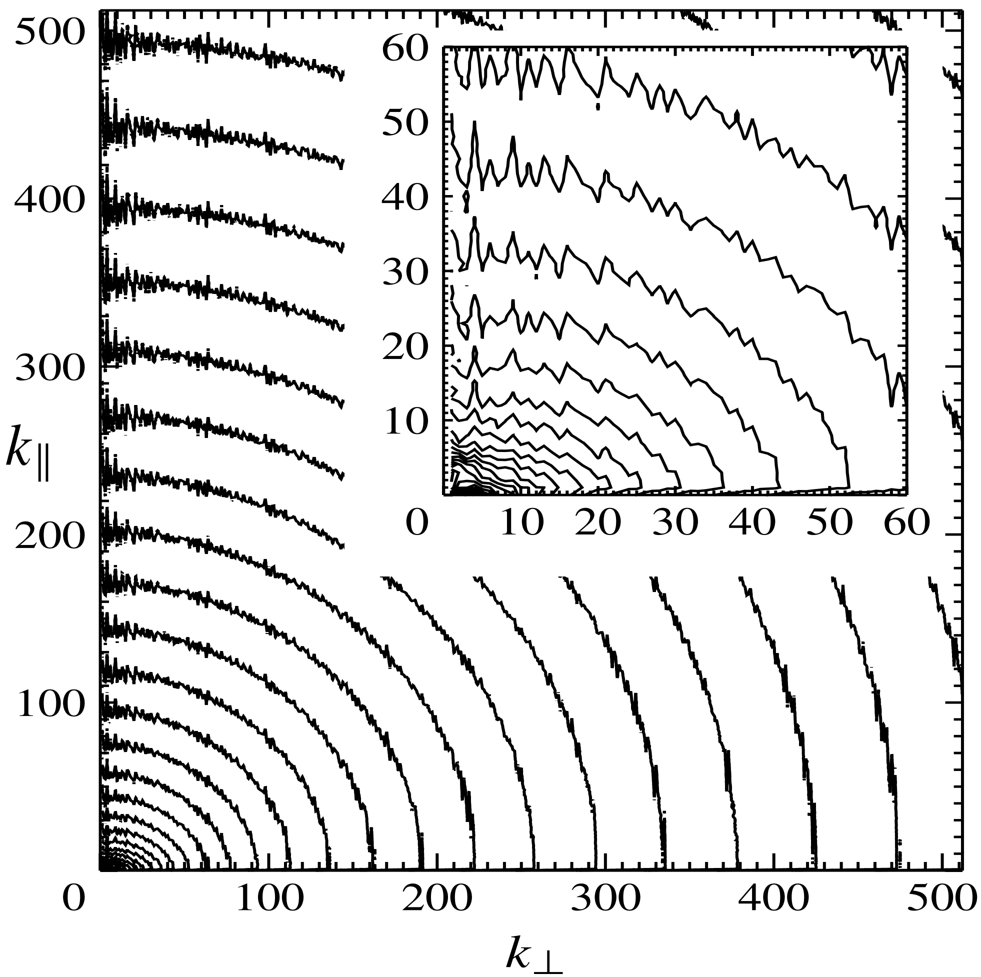
$$k_\Omega = \left(\frac{\Omega^3}{\varepsilon}\right)^{1/2}$$

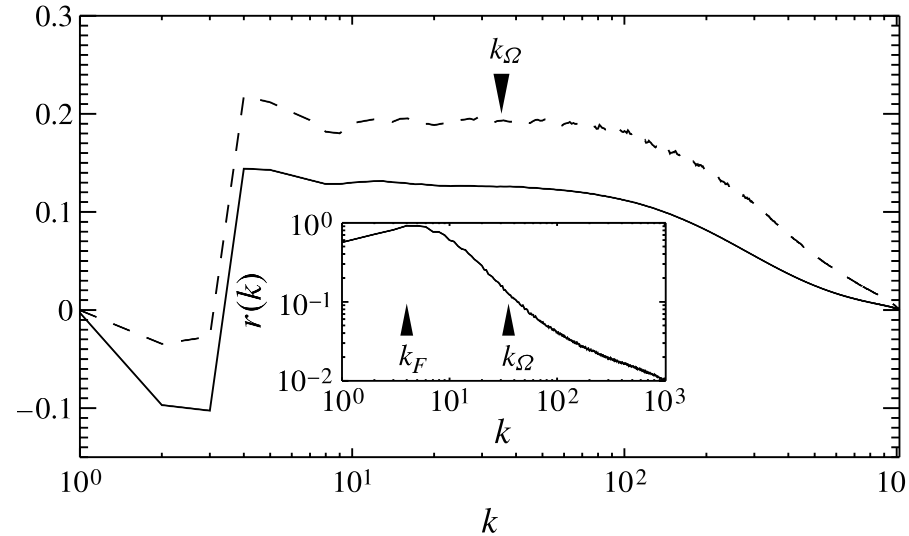
RECOVERY OF ISOTROPY



- 3072^3 simulation of forced turbulence.

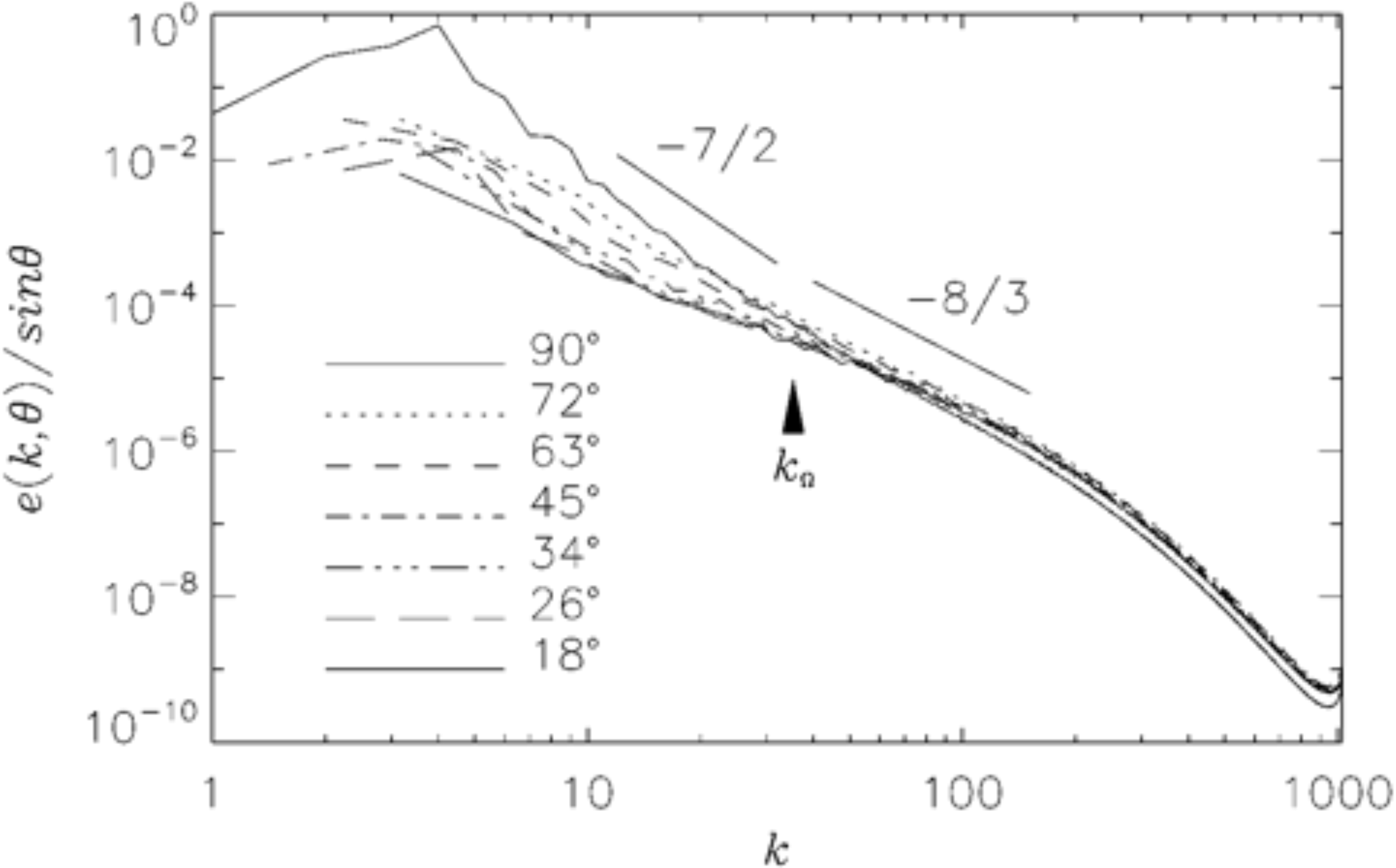
RECOVERY OF ISOTROPY





- 3072^3 simulation of forced rotating turbulence.

RECOVERY OF ISOTROPY



Mininni, Rosenberg, & Pouquet, J. Fluid Mech. **699**, 263 (2012), see also Delache, Cambon and Godeferd (2014)

ENERGY TRANSFER AND TRIADIC INTERACTIONS

- We can decompose the velocity field as

$$\mathbf{u}(\mathbf{k},t) = a_+(\mathbf{k},t)\mathbf{h}_+ + a_-(\mathbf{k},t)\mathbf{h}_-$$

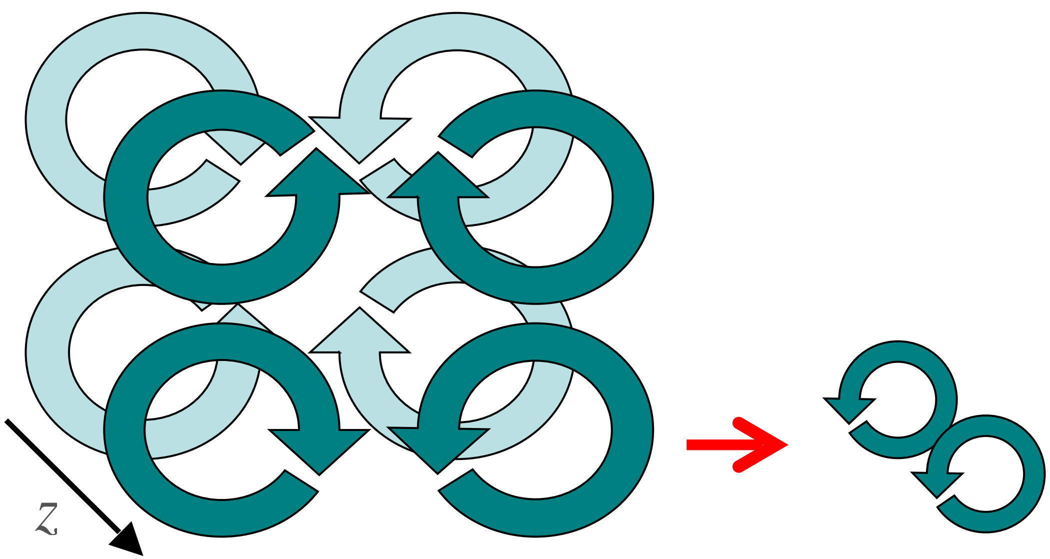
$$a_s(\mathbf{k},t) = A_s(T)e^{i\omega_{\mathbf{k}}t}$$

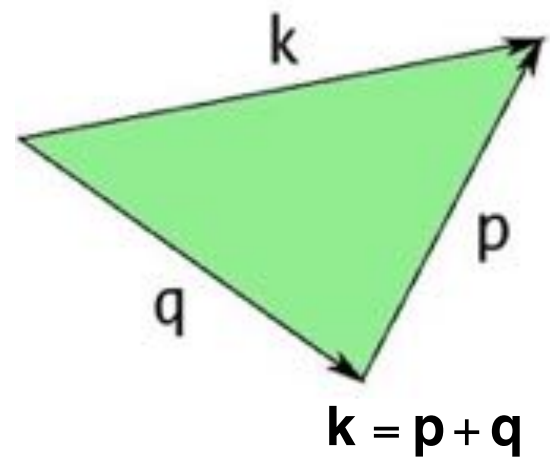
Craya (1958), Herring (1974), Waleffe (1993).

ENERGY TRANSFER AND TRIADIC INTERACTIONS

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{v} + \mathbf{F}$$

$$\Rightarrow \frac{\partial \mathbf{v}_k}{\partial t} = -\int_{p,q} \left[\left(\mathbf{v}_p \cdot \nabla \right) \mathbf{v}_q \right] dp dq - i\mathbf{k} P_k - \nu k^2 \mathbf{v}_k + \mathbf{F}_k$$

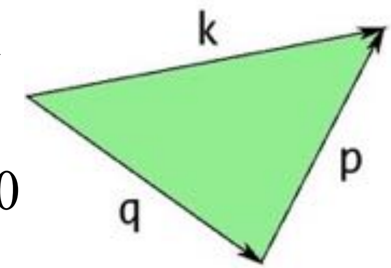




TRIADIC INTERACTIONS IN ROTATING TURBULENCE

- The evolution of the kinetic energy in shells in Fourier space is

$$\frac{\partial \mathbf{v}_k}{\partial t} = -\int_{p,q} \left[\left(\mathbf{v}_p \cdot \nabla \right) \mathbf{v}_q \right] dp\,dq - i\mathbf{k}P_k - \nu k^2 \mathbf{v}_k + \mathbf{F}_k$$



$$\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$$

- In rotating flows we have Rossby waves, that slow down the energy transfer through resonant interactions (Cambon and Jacquin 1989, Cambon, Mansour, and Godeferd 1997, also WT see Galtier 2003):

$$u_k \rightarrow A_{s,k} e^{i\omega_{s,k} t}$$

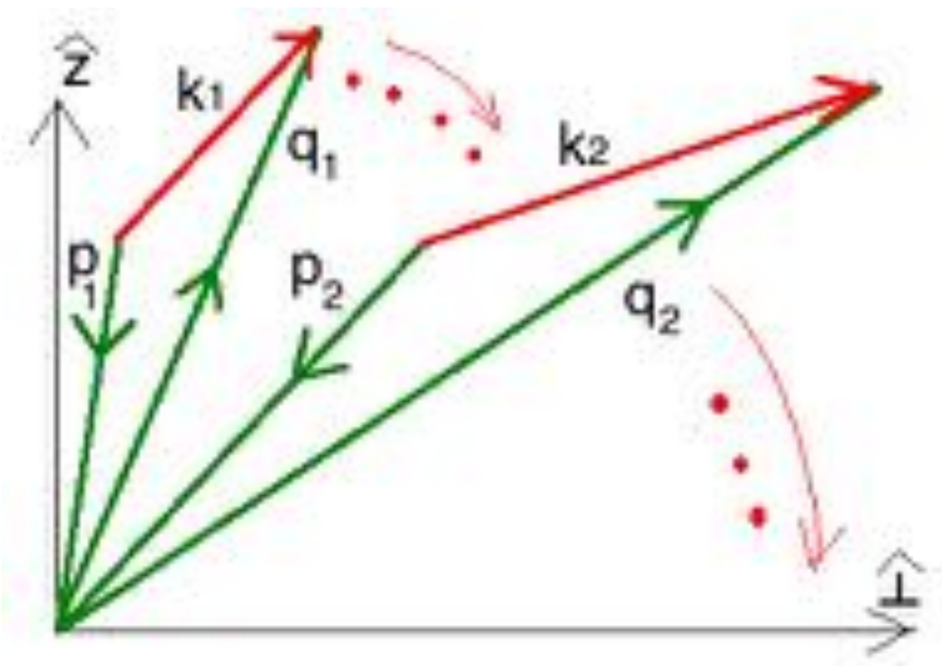
$$\int_{p,q} \left[\left(\mathbf{v}_p \cdot \nabla \right) \mathbf{v}_q \right] dp\,dq \rightarrow \int_{p,q} \left[\left(A_{s,p} \cdot \nabla \right) A_{s,q} \right] e^{i(\omega_{s,k} + \omega_{s,p} + \omega_{s,q})t}$$

$$\omega_{s,k} + \omega_{s,p} + \omega_{s,q} = s_k \frac{k_z}{k} + s_p \frac{p_z}{p} + s_q \frac{q_z}{q} = 0$$

ENERGY TRANSFER AND TRIADIC INTERACTIONS

$$\partial_t a^{s_k}(t) = \mathcal{R}o \sum_{s_p, s_q} \int_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} C_{kpq}^{s_k s_p s_q} a^{s_p^\star} a^{s_q^\star} e^{i(\omega_{s_k}+\omega_{s_p}+\omega_{s_q})t} dp dq$$

$$s_k \frac{k_{||}}{k} + s_p \frac{p_{||}}{p} + s_q \frac{q_{||}}{q} = \mathcal{O}(\mathcal{R}o)$$

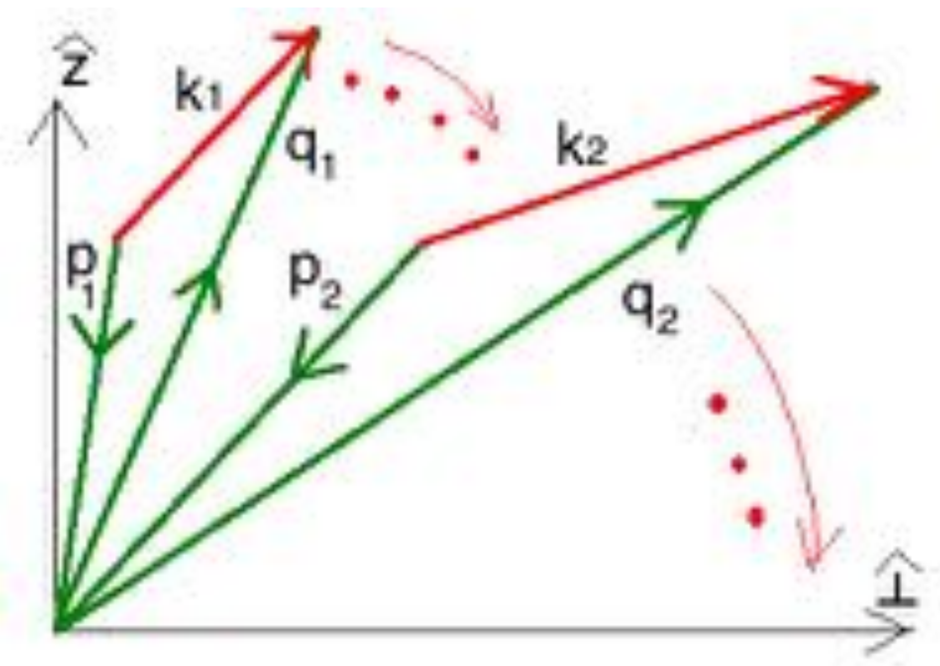


- Instability theorem (Waleffe 1993).
- However, this is not valid for too small values of k_z.
- See Lamriben, Cortet & Moisy 2011 for an experimental study of anisotropic transfer.

PHENOMENOLOGY REVISITED

$$\partial_t a^{s_k}(t) = \mathcal{R}o \sum_{s_p, s_q} \int_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} C_{kpq}^{s_k s_p s_q} a^{s_p^\star} a^{s_q^\star} \, e^{i(\omega_{s_k}+\omega_{s_p}+\omega_{s_q})t} dp dq$$

$$s_k \frac{k_{||}}{k} + s_p \frac{p_{||}}{p} + s_q \frac{q_{||}}{q} = \mathcal{O}(\mathcal{R}o)$$

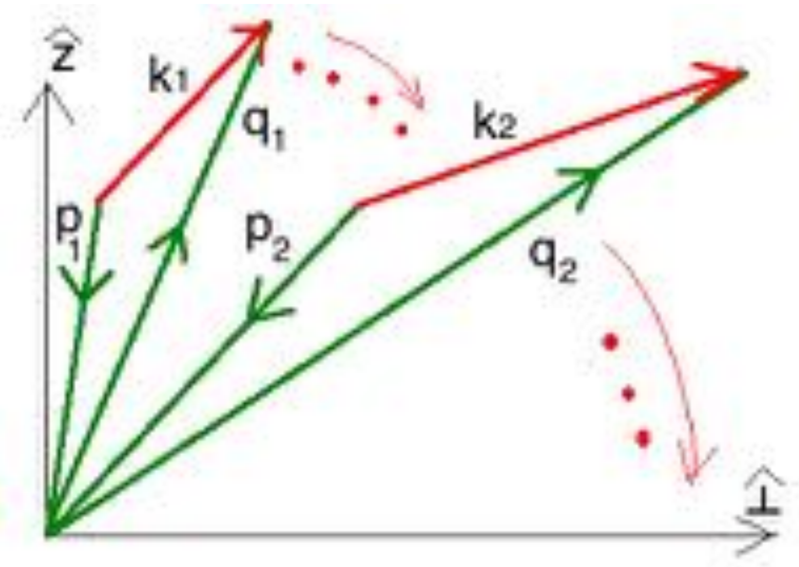


The rate of energy transfer can be estimated as

$$\epsilon \sim \left(\frac{u_\ell}{\ell_\perp \Omega}\right)\left(\frac{u_\ell^3}{\ell_\perp}\right) = \frac{u_\ell^4}{\ell_\perp^2 \Omega}$$

ENERGY TRANSFER AND TRIADIC INTERACTIONS

$$s_k \frac{k_{||}}{k} + s_p \frac{p_{||}}{p} + s_q \frac{q_{||}}{q} = \mathcal{O}(\mathcal{R}o)$$



- To transfer energy to 2D modes, near-resonant and non-resonant interactions are needed.
- Smith & Lee (2005): Truncated simulations with only some interactions preserved. Near-resonant interactions are needed to reproduce the quasi-two dimensionalisation of the flow
- Alexakis (2015): Analysis of a large numerical dataset. The dynamics of the 2D modes can only be captured if near-resonant and non-resonant interactions are taken into account.

ENERGY TRANSFER AND TRIADIC INTERACTIONS

- From the momentum equation, we can derive an equation for the evolution of the correlation functions:

$$\frac{\partial}{\partial t} \left\langle \mathbf{u'}^*_{\mathbf{k}} \cdot \mathbf{u}_{\mathbf{k}} \right\rangle_{t'} = -i \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \left\langle \mathbf{u'}^*_{\mathbf{k}} \cdot (\mathbf{u}_{\mathbf{p}} \cdot \mathbf{q}) \mathbf{u}_{\mathbf{q}} \right\rangle_{t'}$$

- The term on the r.h.s. is a triple correlation associated with triadic interactions.

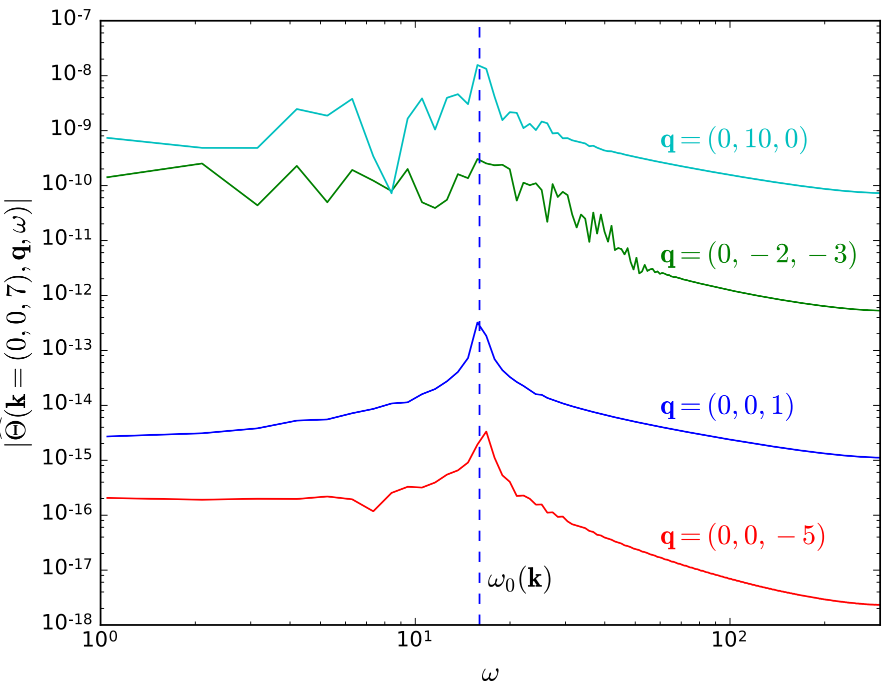
$$\Theta(\mathbf{k}, \mathbf{p}, \mathbf{q}, \tau) = \langle \mathbf{u}^*_{\mathbf{k}}(t') \cdot [\mathbf{u}_{\mathbf{p}}(t'+\tau) \cdot \mathbf{q}]\, \mathbf{u}_{\mathbf{q}}(t'+\tau) \rangle_{t'}$$

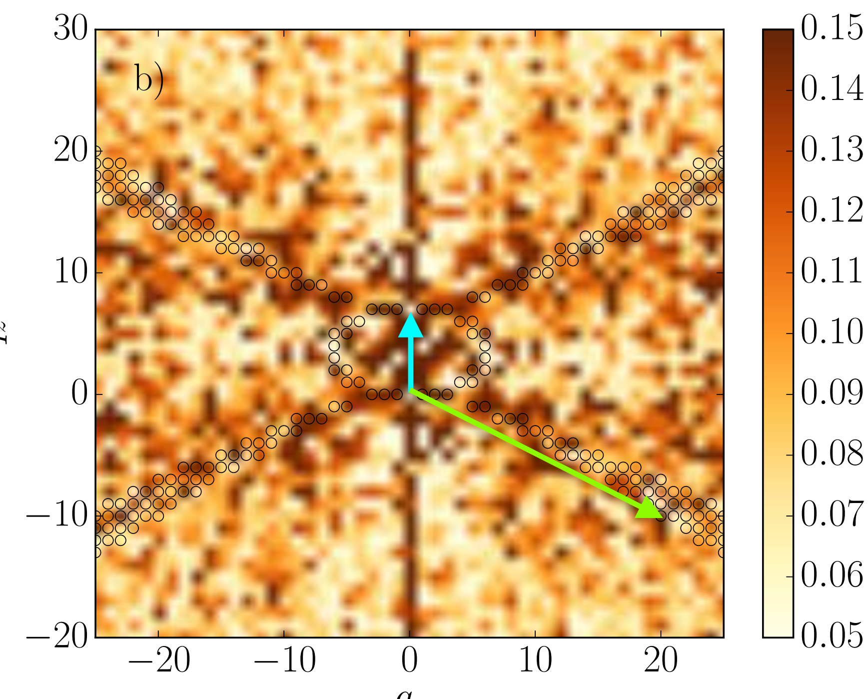
- For pure wave modes, the Fourier transform of the triple correlation is perfectly tuned in the wave frequency (i.e., in the resonance):

$$\begin{aligned} \widehat{\Theta}(\mathbf{k}, \mathbf{q}, \mathbf{p}, \omega) &= \int_{-\infty}^{\infty} e^{i\omega\tau} \left\langle \mathbf{u}^*_{\mathbf{k}}(t') \cdot [\mathbf{u}_{\mathbf{p}}(t'+\tau) \cdot \mathbf{q}]\, \mathbf{u}_{\mathbf{q}}(t'+\tau) \right\rangle_{t'} \mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} e^{i(\omega+\omega_{\mathbf{p}}+\omega_{\mathbf{q}})\tau} \left\langle \mathbf{U}^*_{\mathbf{k}} \cdot (\mathbf{U}_{\mathbf{p}} \cdot \mathbf{q})\, \mathbf{U}_{\mathbf{q}}\, e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{p}}-\omega_{\mathbf{q}})t'} \right\rangle_{t'} \mathrm{d}\tau \\ &= \mathbf{U}^*_{\mathbf{k}} \cdot (\mathbf{U}_{\mathbf{p}} \cdot \mathbf{q}) \mathbf{U}_{\mathbf{q}}\, \delta(\omega - \omega_{\mathbf{k}}). \end{aligned}$$

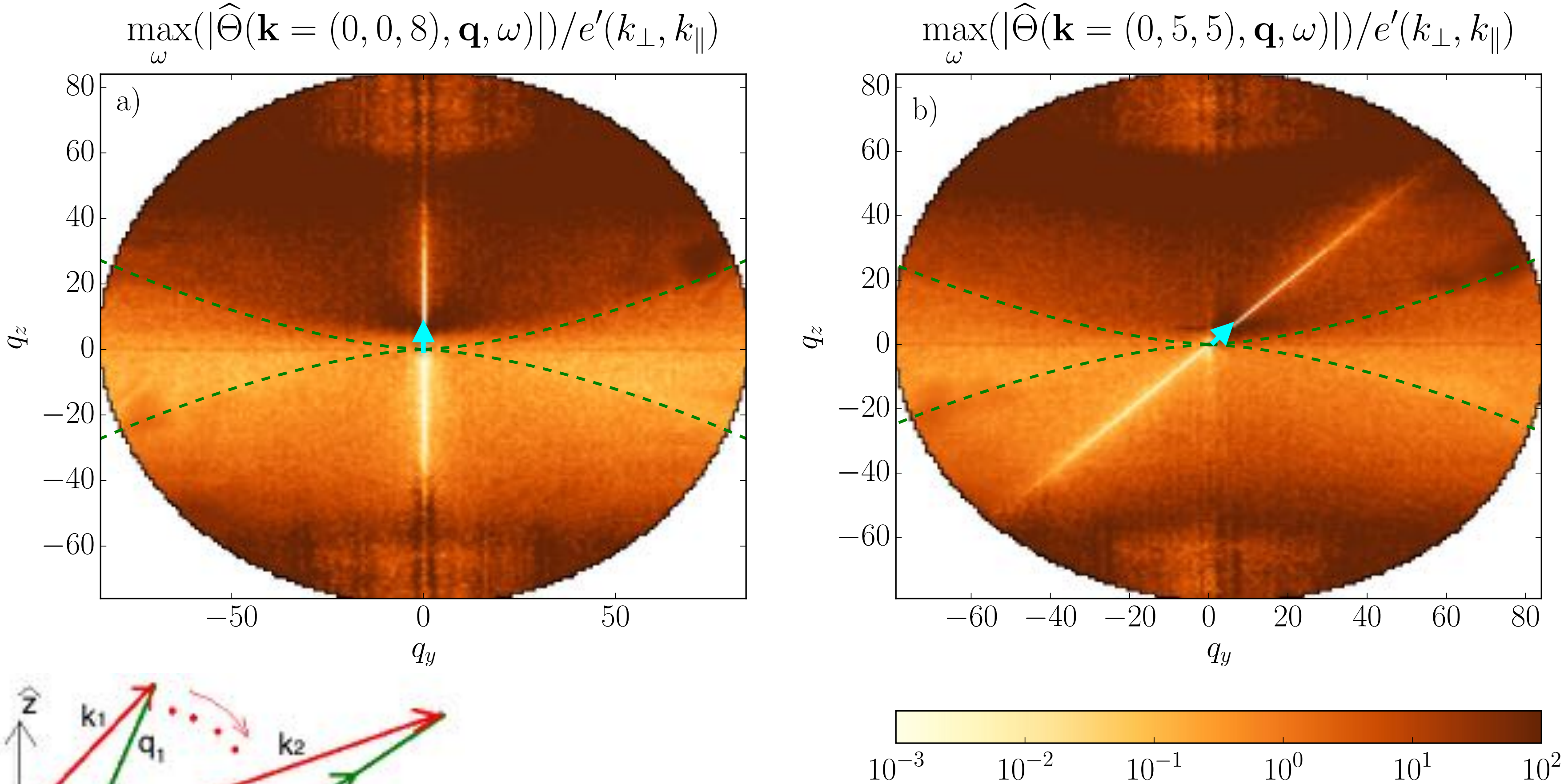
ENERGY TRANSFER AND TRIADIC INTERACTIONS

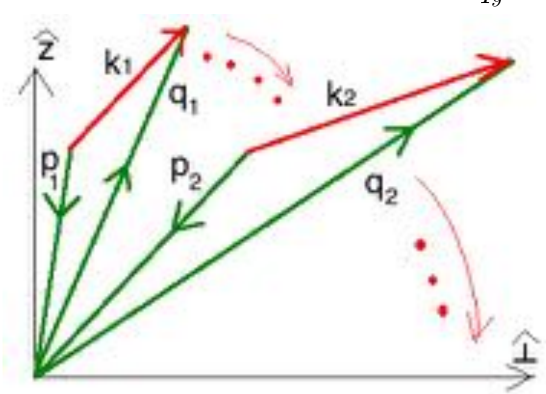
$$\begin{aligned}
\widehat{\Theta}(\mathbf{k},\mathbf{q},\mathbf{p},\omega) &= \int_{-\infty}^{\infty} e^{i\omega\tau} \left\langle \mathbf{u}_{\mathbf{k}}^{*}(t') \cdot [\mathbf{u}_{\mathbf{p}}(t'+\tau)\cdot\mathbf{q}]\, \mathbf{u}_{\mathbf{q}}(t'+\tau) \right\rangle_{t'} \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} e^{i(\omega+\omega_{\mathbf{p}}+\omega_{\mathbf{q}})\tau} \left\langle \mathbf{U}_{\mathbf{k}}^{*} \cdot (\mathbf{U}_{\mathbf{p}}\cdot\mathbf{q})\, \mathbf{U}_{\mathbf{q}}\, e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{p}}-\omega_{\mathbf{q}})t'} \right\rangle_{t'} \mathrm{d}\tau \\
&= \mathbf{U}_{\mathbf{k}}^{*} \cdot (\mathbf{U}_{\mathbf{p}}\cdot\mathbf{q})\mathbf{U}_{\mathbf{q}}\, \delta(\omega-\omega_{\mathbf{k}}).
\end{aligned}$$





ENERGY TRANSFER AND TRIADIC INTERACTIONS





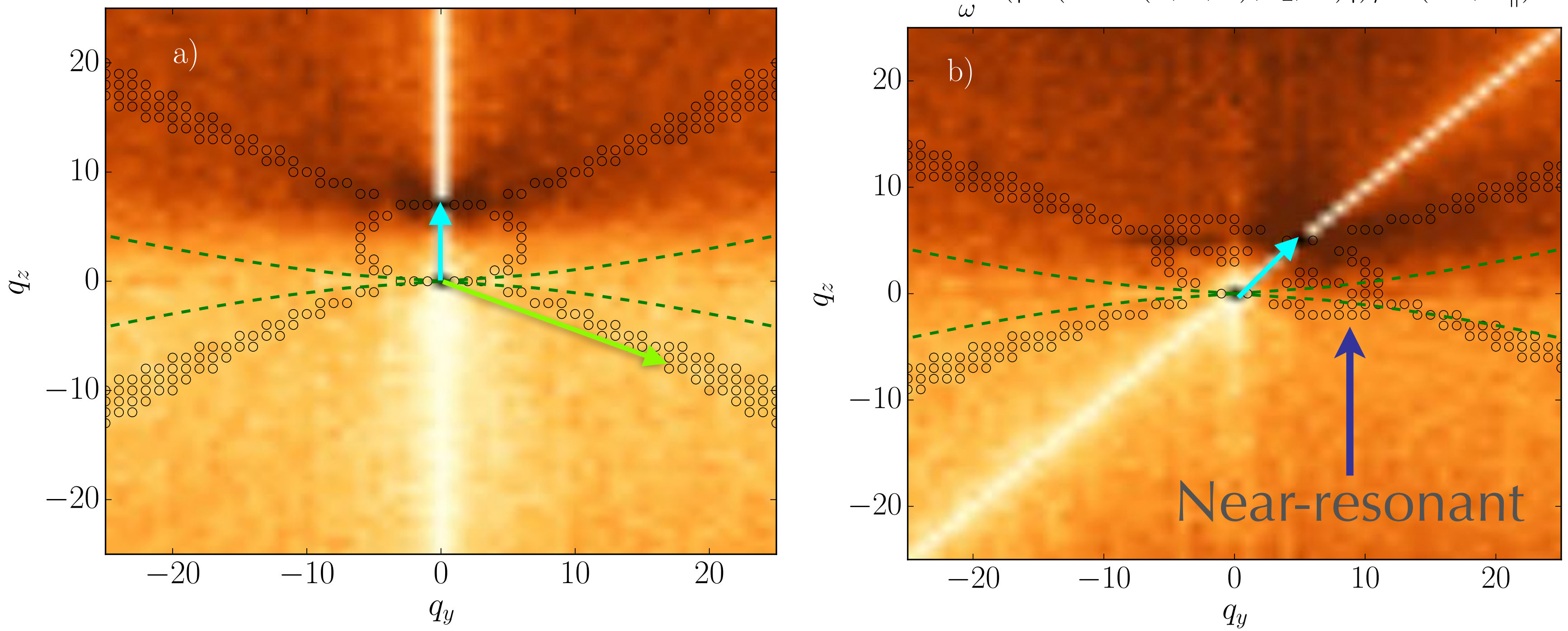
$$\Theta(\mathbf{k},\mathbf{p},\mathbf{q},\tau) = \langle \mathbf{u}^*_{\mathbf{k}}(t') \cdot [\mathbf{u}_{\mathbf{p}}(t'+\tau)\cdot\mathbf{q}]\,\mathbf{u}_{\mathbf{q}}(t'+\tau)\rangle_{t'}$$

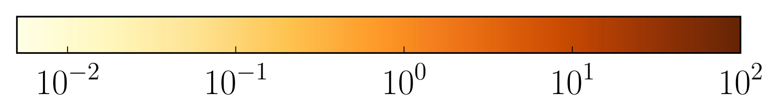
Clark di Leoni & Mininni, arXiv (2016)

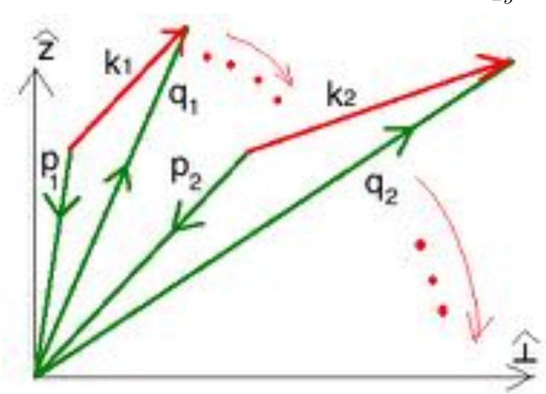
ENERGY TRANSFER AND TRIADIC INTERACTIONS

$\max_{\omega}(|\widehat{\Theta}(\mathbf{k}=(0,0,8),\mathbf{q},\omega)|)/e'(k_\perp,k_\parallel)$

$\max_{\omega}(|\widehat{\Theta}(\mathbf{k}=(0,5,5),\mathbf{q},\omega)|)/e'(k_\perp,k_\parallel)$







$$\Theta(\mathbf{k},\mathbf{p},\mathbf{q},\tau)=\langle \mathbf{u}^*_{\mathbf{k}}(t')\cdot[\mathbf{u}_{\mathbf{p}}(t'+\tau)\cdot\mathbf{q}]\,\mathbf{u}_{\mathbf{q}}(t'+\tau)\rangle_{t'}$$

Clark di Leoni & Mininni, arXiv (2016)

INVERSE CASCADE

- Once the energy reaches the 2D modes, it can develop an inverse transfer towards large scales.
- Returning to the 2D+3D decomposition:

$$\mathbf{u}(\mathbf{k}) = \begin{cases} \mathbf{u}_{3D}(\mathbf{k}) & \text{if } \mathbf{k} \in W_k \\ \mathbf{u}_\perp(\mathbf{k}_\perp) + w(\mathbf{k}_\perp)\hat{z} & \text{if } \mathbf{k} \in V_k \end{cases}$$

$$W_k := \{\mathbf{k} \text{ s.t. } |\mathbf{k}| \neq 0 \text{ and } k_{||} \neq 0\}$$

$$V_k := \{\mathbf{k} \text{ s.t. } |\mathbf{k}| \neq 0 \text{ and } k_{||} = 0\}$$

- We can write equations for the energy in these modes:

$$d_t E_{3D} = \Pi_{2D\to 3D} - \Pi_{3D} + \epsilon_{3D},$$

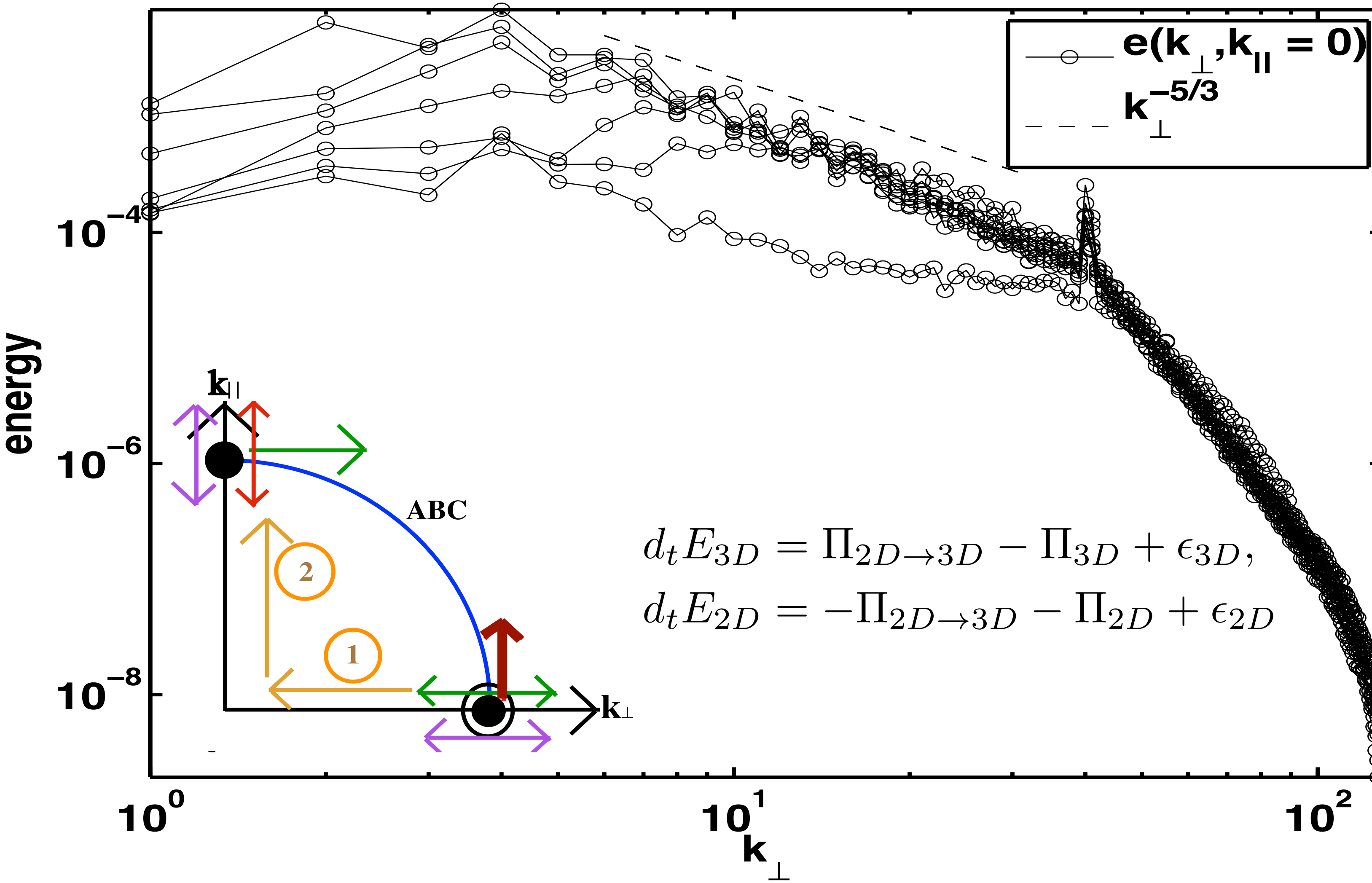
$$d_t E_{2D} = -\Pi_{2D\to 3D} - \Pi_{2D} + \epsilon_{2D}$$

- If the coupling between 2D and 3D modes goes to zero for zero Ro:

$$\frac{\partial \overline{\mathbf{u}}_\perp}{\partial t} + \overline{\mathbf{u}}_\perp \cdot \boldsymbol{\nabla} \overline{\mathbf{u}}_\perp = -\boldsymbol{\nabla}\overline{\mathcal{P}} + \nu \nabla^2 \overline{\mathbf{u}}_\perp$$

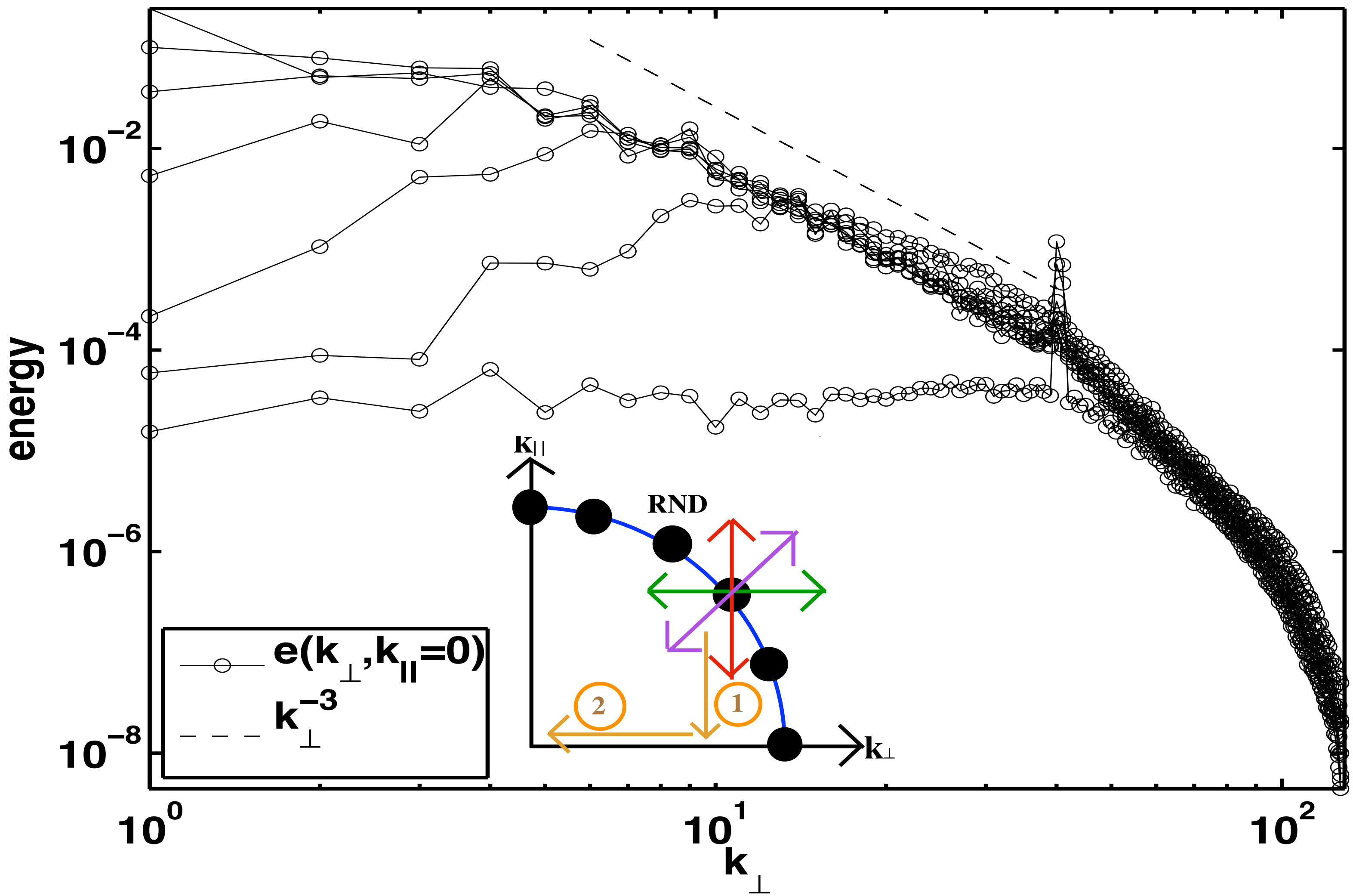
(see however Alexakis 2015, Gallet 2015).

INVERSE CASCADE



Sen, Mininni, Rosenberg, & Pouquet, PRE **86**, 036319 (2012), also Campagne et al. (2015).

INVERSE CASCADE



Sen, Mininni, Rosenberg, & Pouquet, PRE **86**, 036319 (2012), also Campagne et al. (2015).

HELICITY AS AN INVARIANT OF 3D EULER

- Euler equations for an ideal, incompressible fluid with uniform density (1757):

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p$$

- The equations can be written as

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\omega\times\mathbf{u}\right)=-\nabla p'$$

with $\boldsymbol{\omega}=\nabla\times\mathbf{u}$

- Note that when $\boldsymbol{\omega}\times\mathbf{u}=0$ the non-linear term becomes zero.

HELICAL FLOWS

$$H = \int \boldsymbol{\omega} \cdot \mathbf{u} \, dV$$

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

- When maximal, $\boldsymbol{\omega} \times \mathbf{u} = 0$
- Helicity is thus associated with corkscrew motions.
- As the non-linear term in the momentum equation becomes zero or negligible, helical flows are very stable.

HELICITY WAS DISCOVERED “RECENTLY”

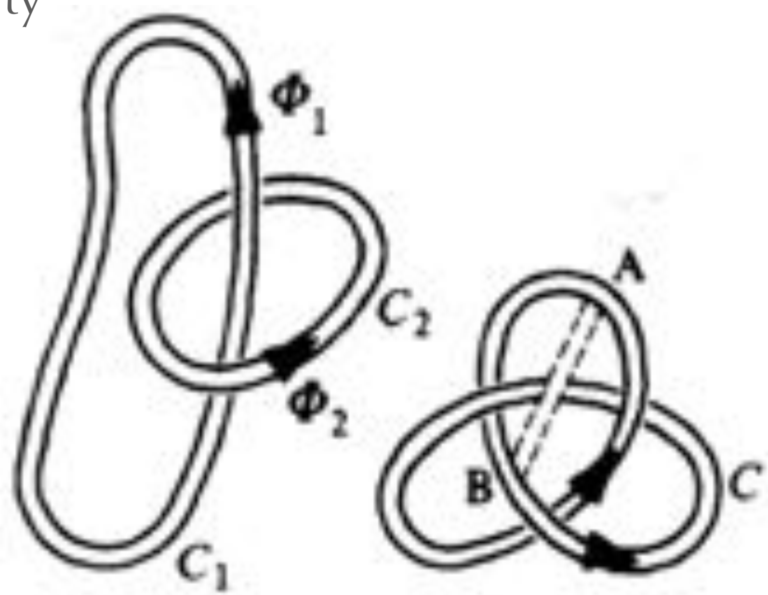
- In 1958 Woltjer introduces the magnetic helicity (later studied by Chandrasekhar and Kendall):

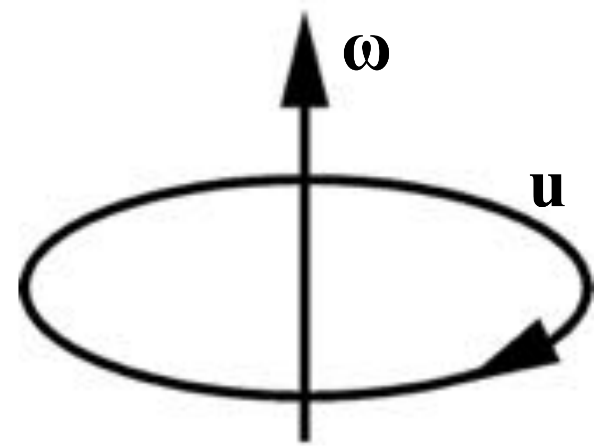
$$H_m = \int \mathbf{B}\cdot\mathbf{A}\,dV \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

- In 1967, Moffatt finds its hydrodynamic equivalent:

$$H = \int \boldsymbol{\omega}\cdot\mathbf{u}\,dV \qquad \boldsymbol{\omega} = \nabla\times\mathbf{u}$$

- Helicity is zero for 2D flows, and it is a conserved quantity in 3D hydrodynamics (without and with rotation).
- Helicity measures the structural complexity of the flow: it is proportional to the number of links in the field lines.
- What is the role of helicity in atmospheric, geophysical, and astrophysical flows?

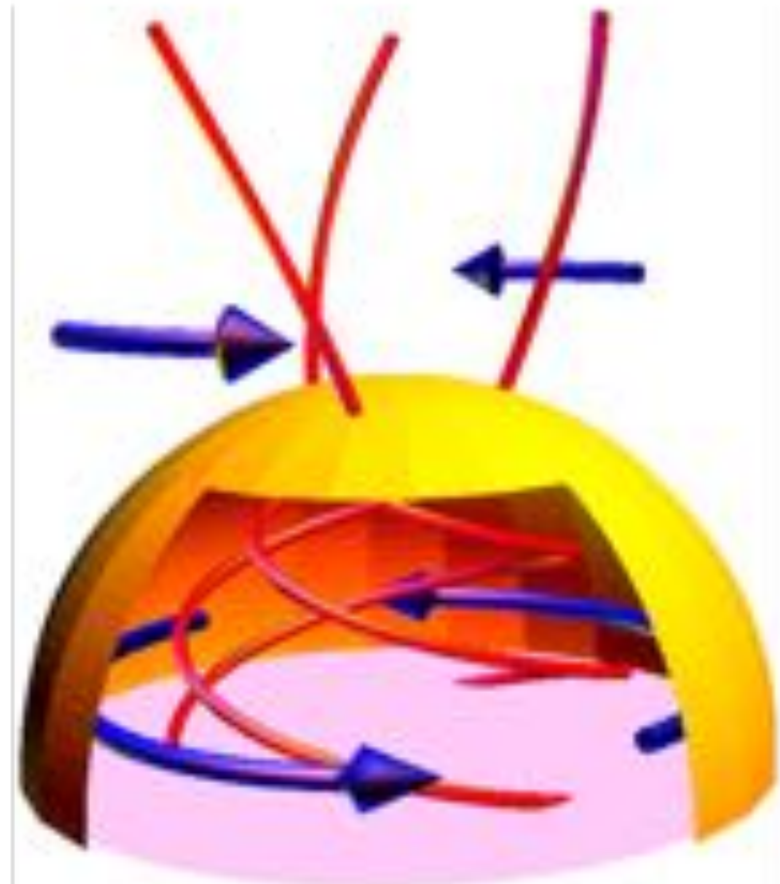




THE ROLE OF HELICITY

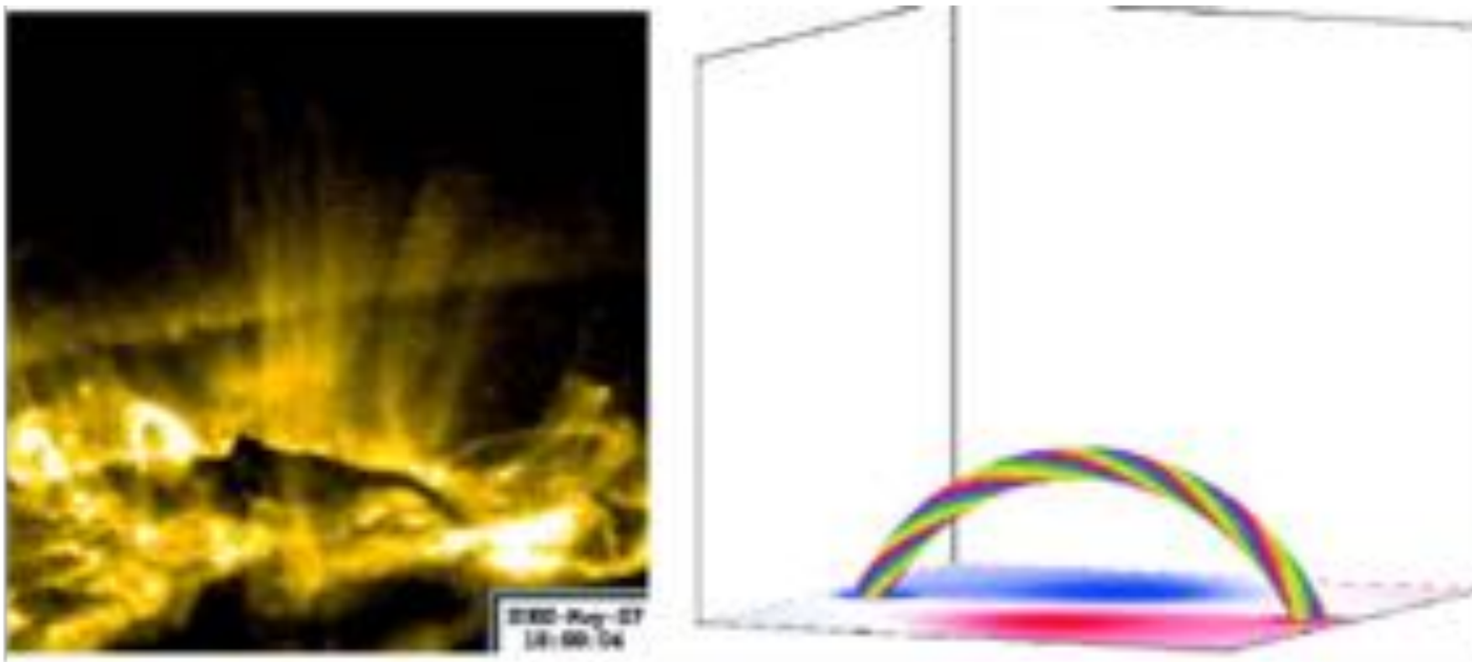
Helical flows are relevant for many applications:

- Solar and geophysical dynamo: helical flows are known to sustain large-scale dynamo action (Parker 1955, Pouquet et al. 1976, Krause & Rädler 1986).
- Helical velocity fields result in the "alpha-effect", and in the generation of magnetic fields by self-induction.
- The large-scale magnetic fields generated by this mechanism are helical.
- The mechanism is also relevant in the presence of kinetic effects (Mininni, Gómez & Mahajan 2003)



Berger (1999)

THE ROLE OF HELICITY



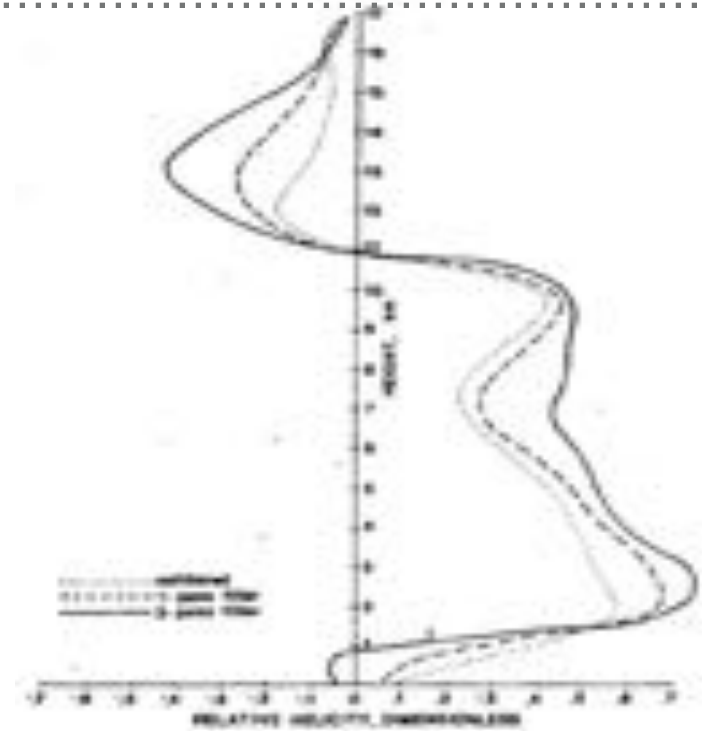
Török & Kliem (2005)

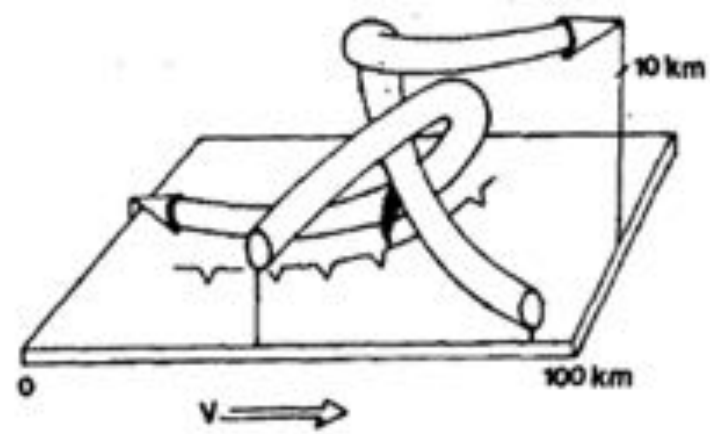
- Helical magnetic fields are observed in the solar wind and the magnetosphere.
- Magnetic fields can be reconstructed from observations of eruptions (e.g., from TRACE), using minimization methods to obtain force-free fields (Titov & Demoulin).

THE ROLE OF HELICITY

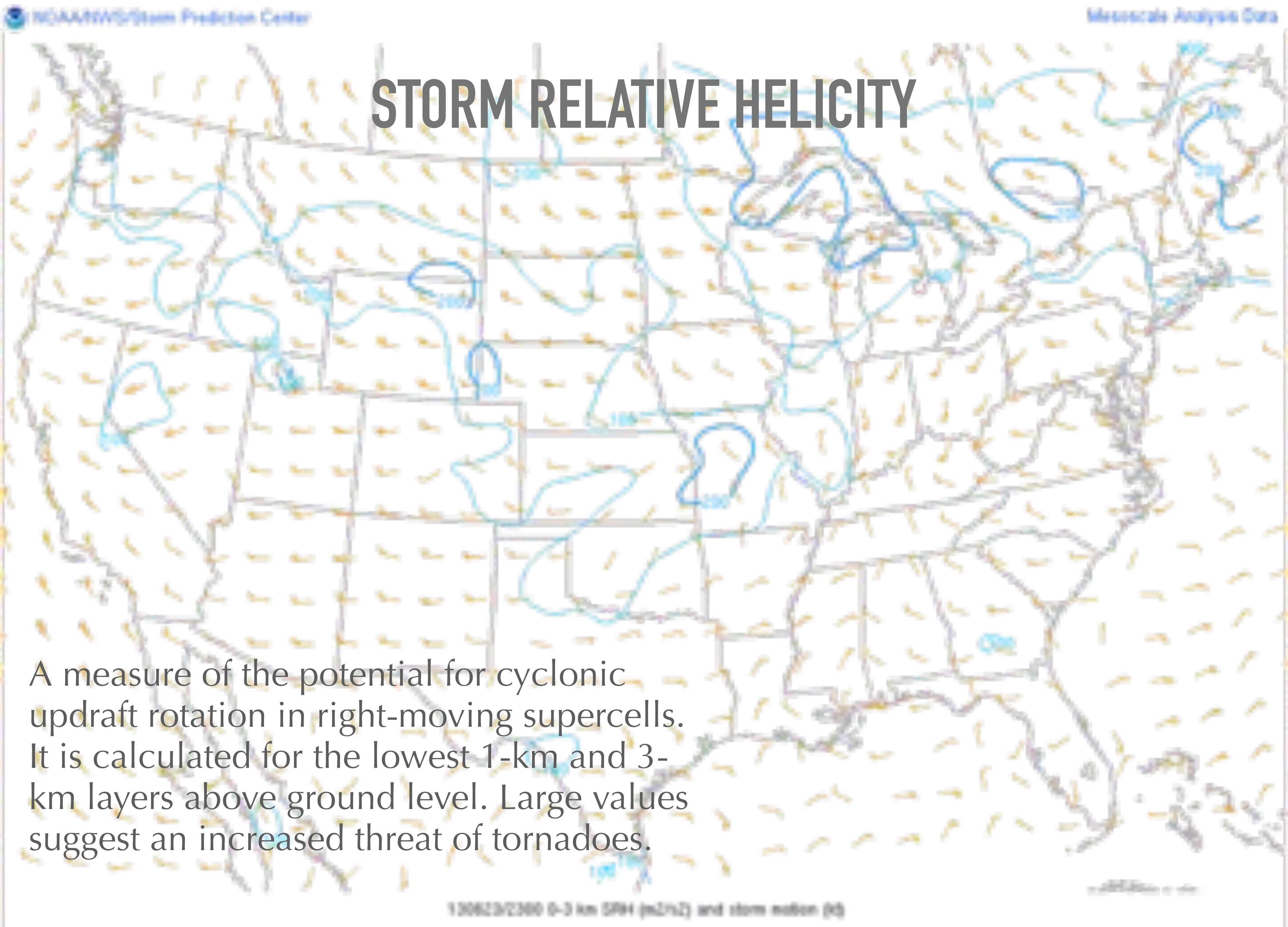
Helical flows are relevant for many applications:

- Atmospheric flows: Lilly (1986) speculated that rotating convective supercell storms are more stable because flows are helical.
- Some authors claim that helicity may play a role in the self-organization of the flow leading to formation of tornadoes (Montgomery 2006, Levina 2013).
- Indices based on helicity are used for forecasting purposes.



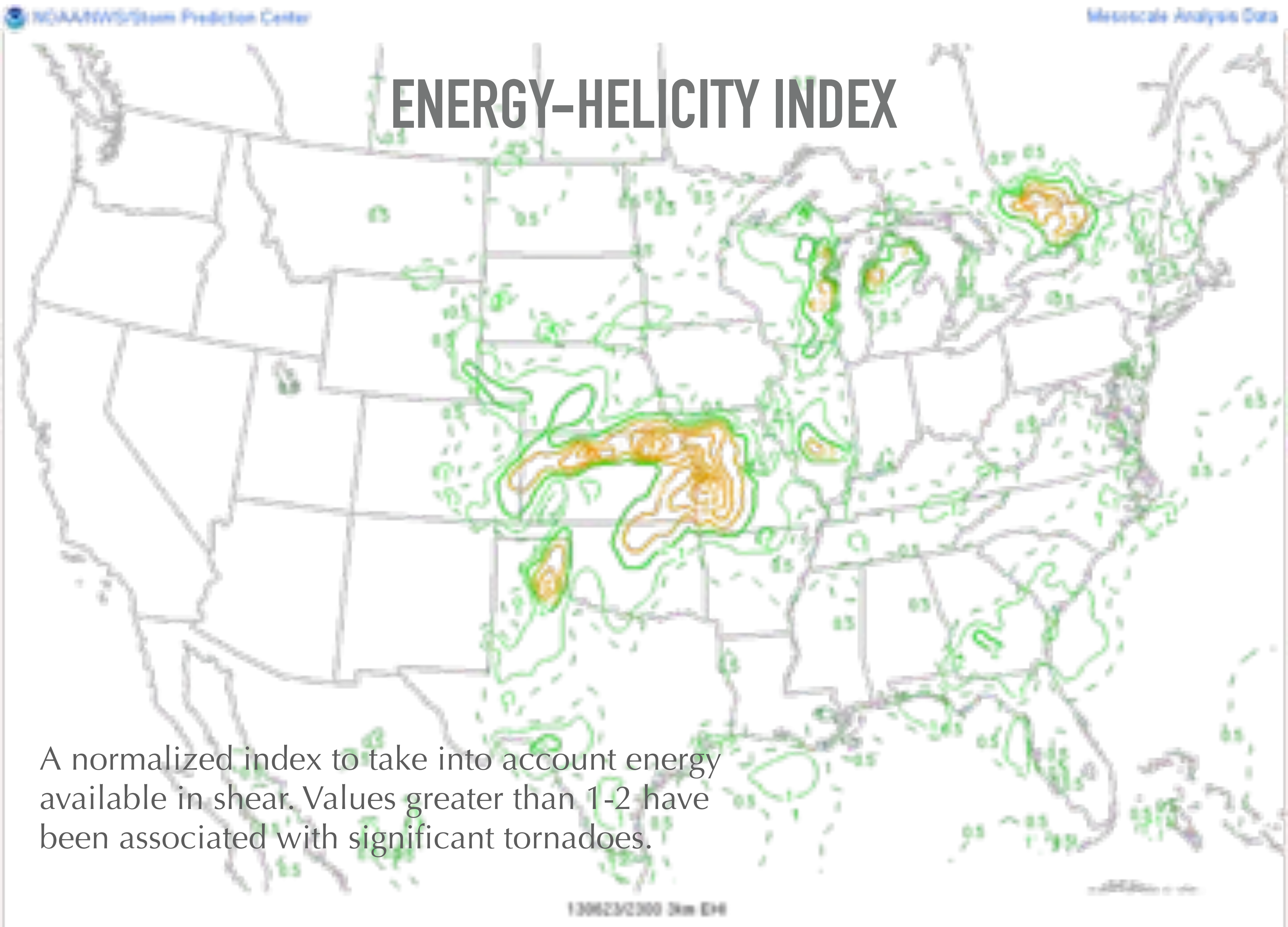


STORM RELATIVE HELICITY



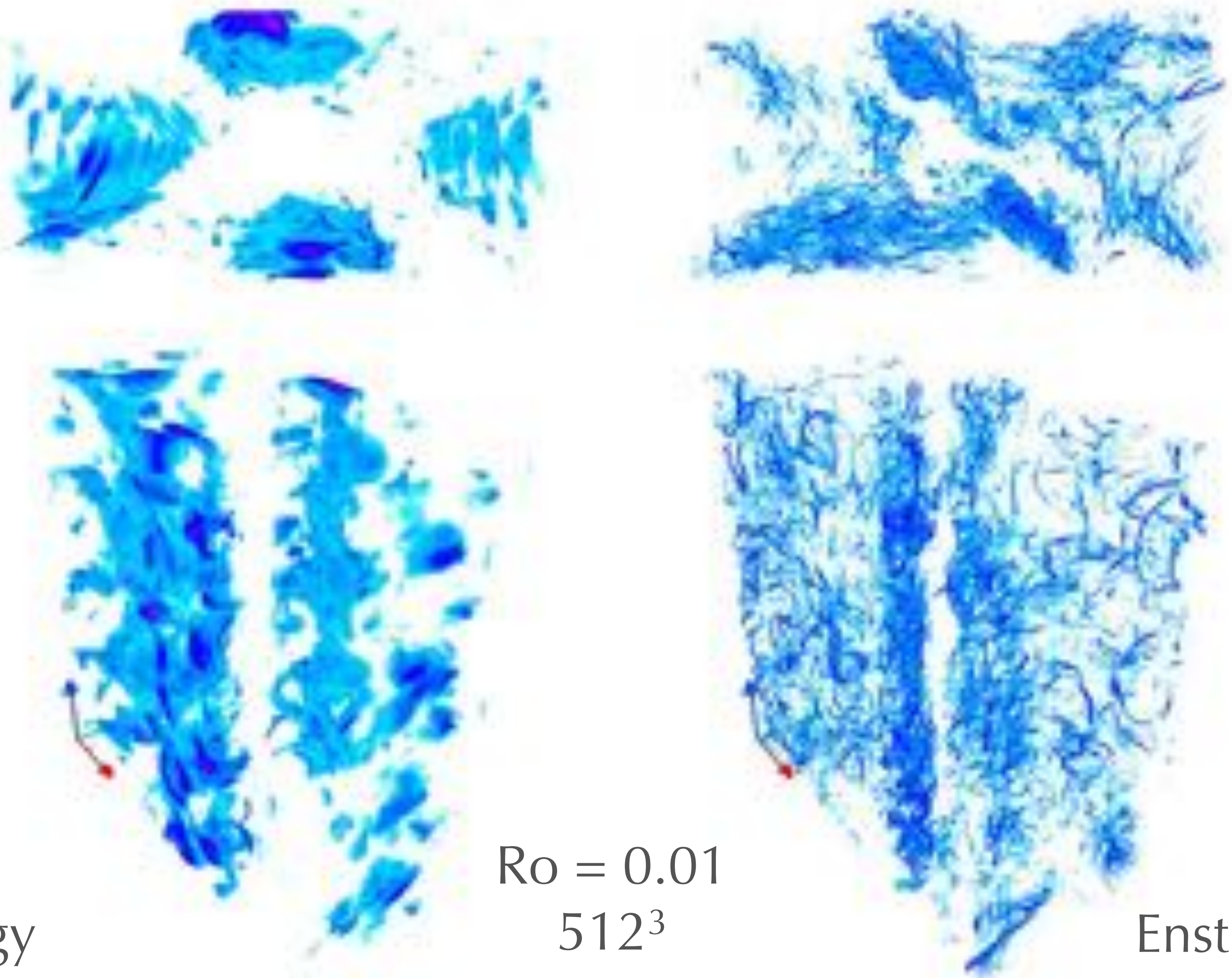
A measure of the potential for cyclonic updraft rotation in right-moving supercells. It is calculated for the lowest 1-km and 3-km layers above ground level. Large values suggest an increased threat of tornadoes.

ENERGY-HELICITY INDEX



A normalized index to take into account energy available in shear. Values greater than 1-2 have been associated with significant tornadoes.

NON-HELICAL ROTATING TURBULENCE

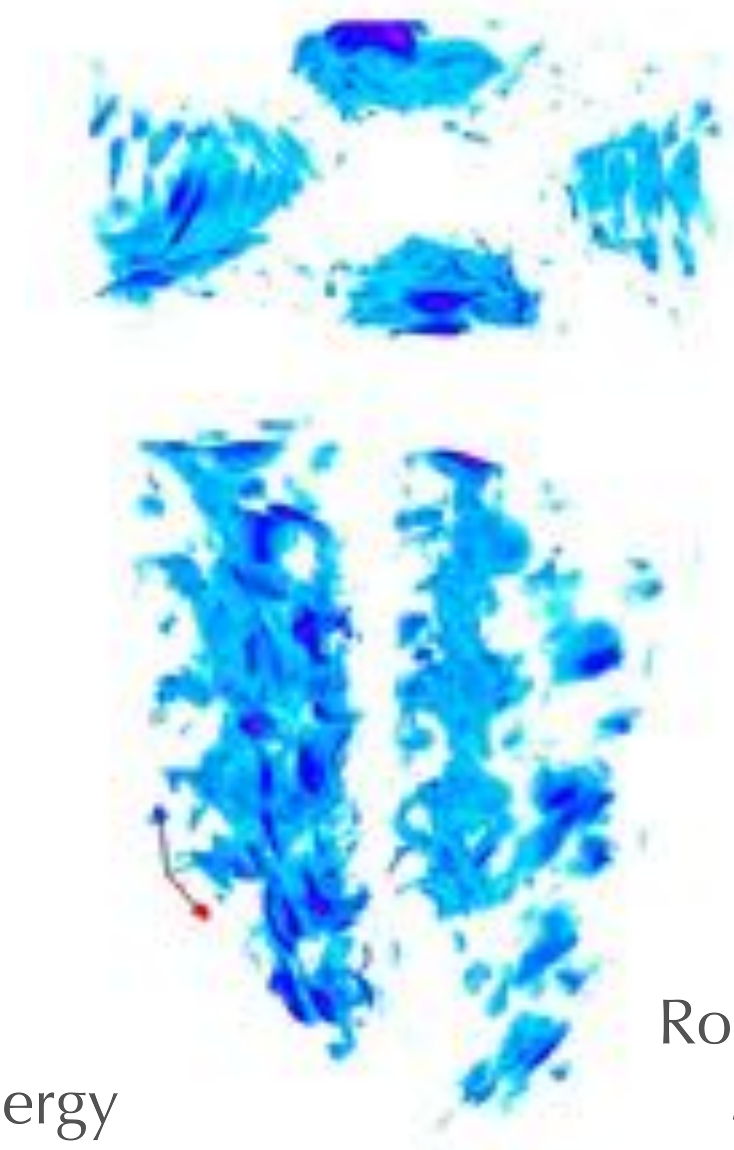


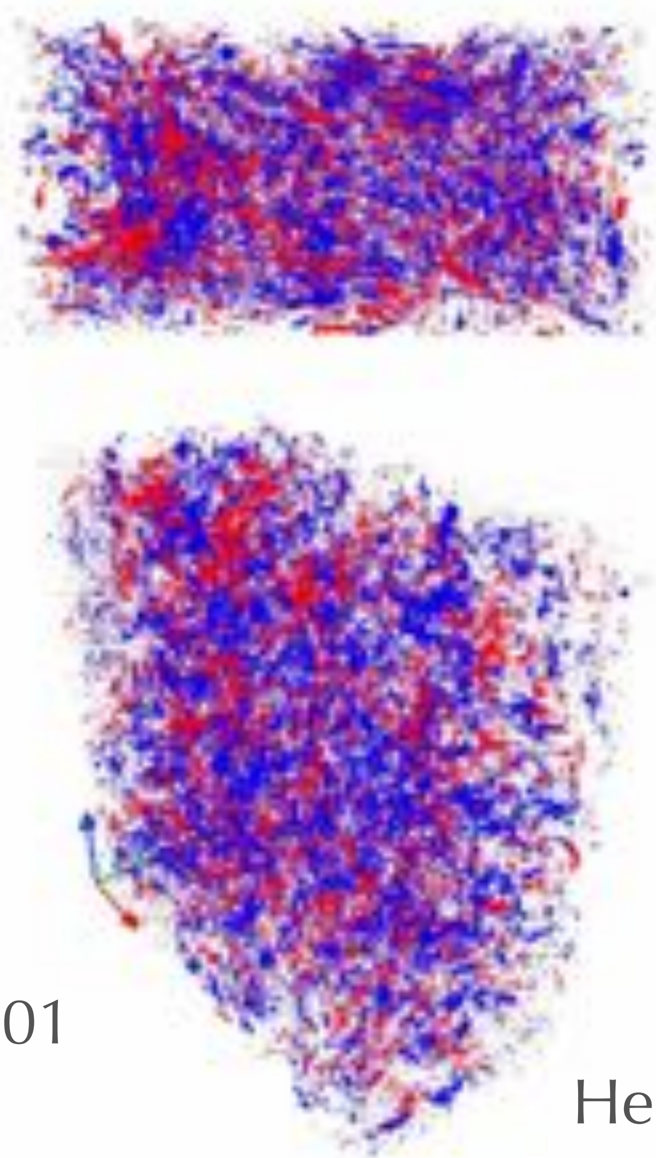
Energy

Ro = 0.01
512^3

Enstrophy

NON-HELICAL ROTATING TURBULENCE





Ro = 0.01

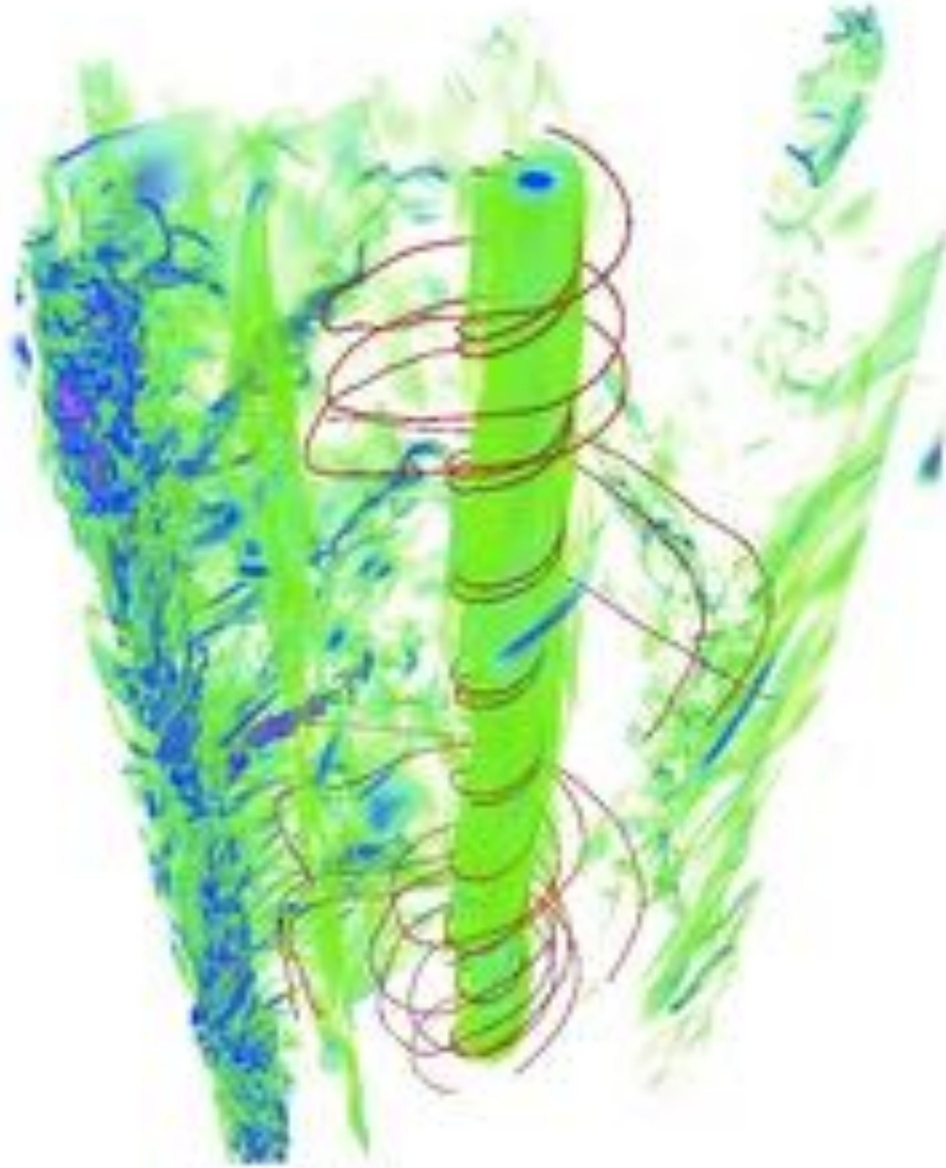
512^3

Energy

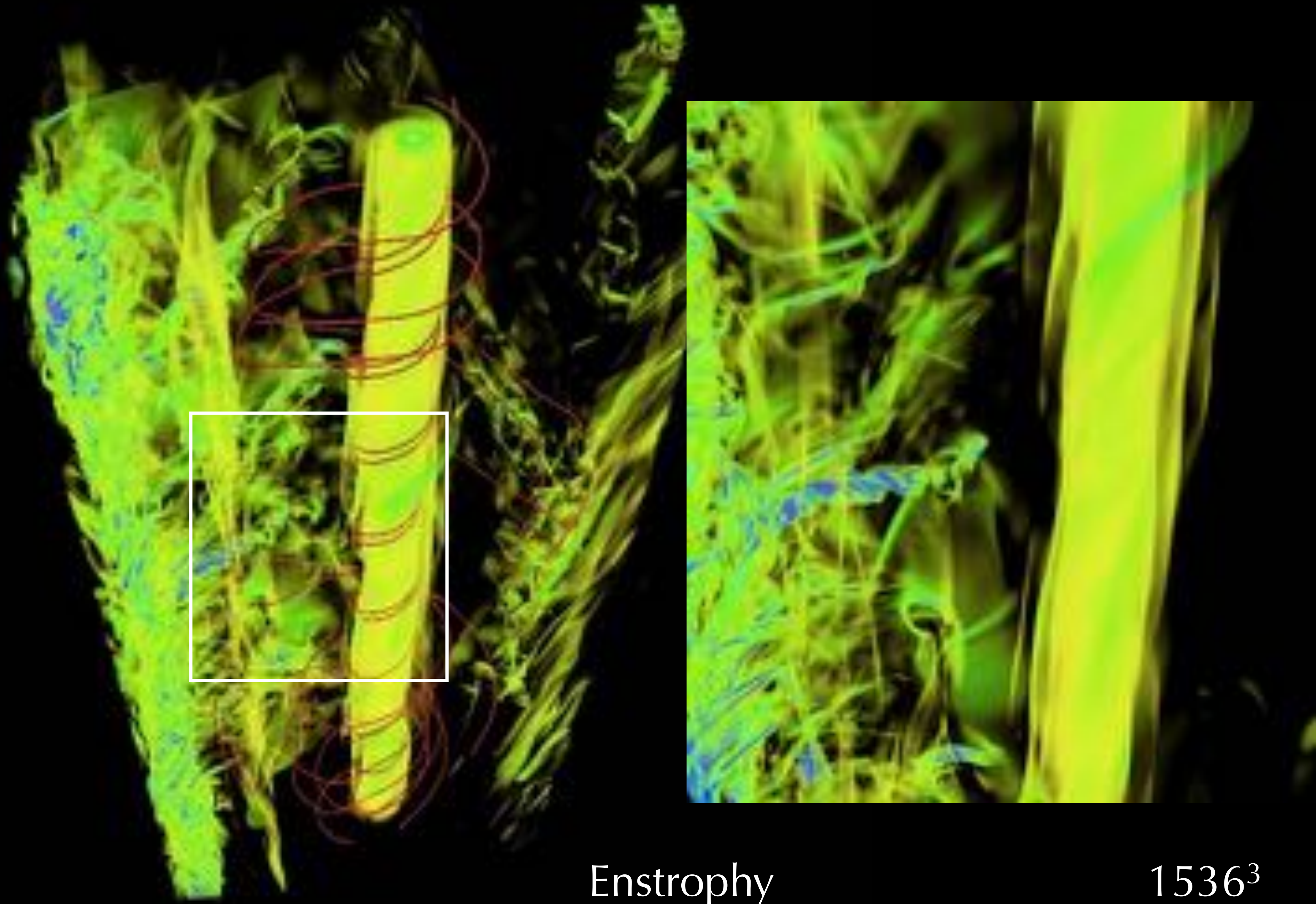
Helicity

HELICAL ROTATING TURBULENCE

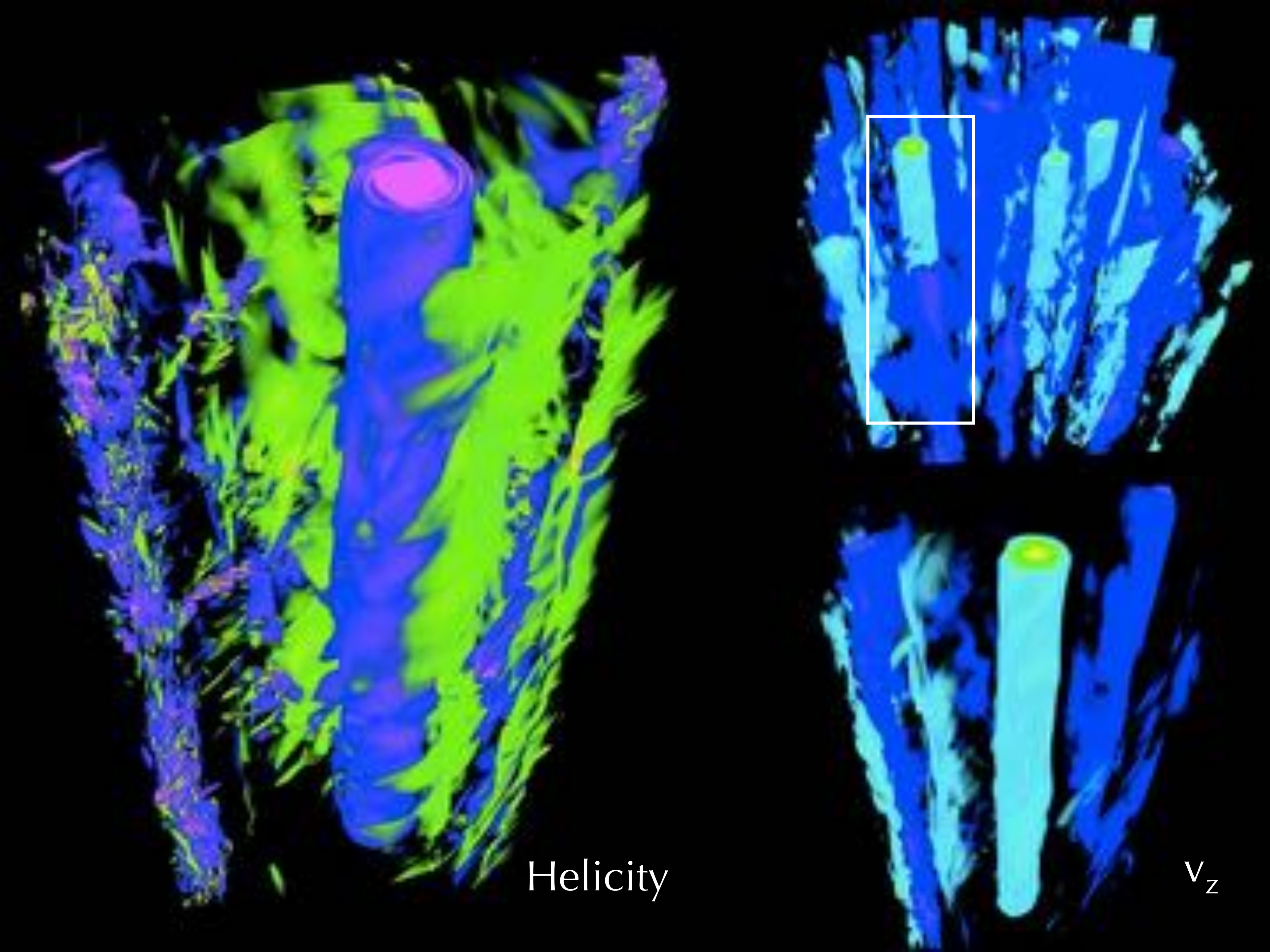
- 512^3 to 3072^3 spatial resolutions.
- Re up to 10000, Ro down to 0.06.
- Laminar column-like structures develop in the flow.
- Structures are helical and stable.

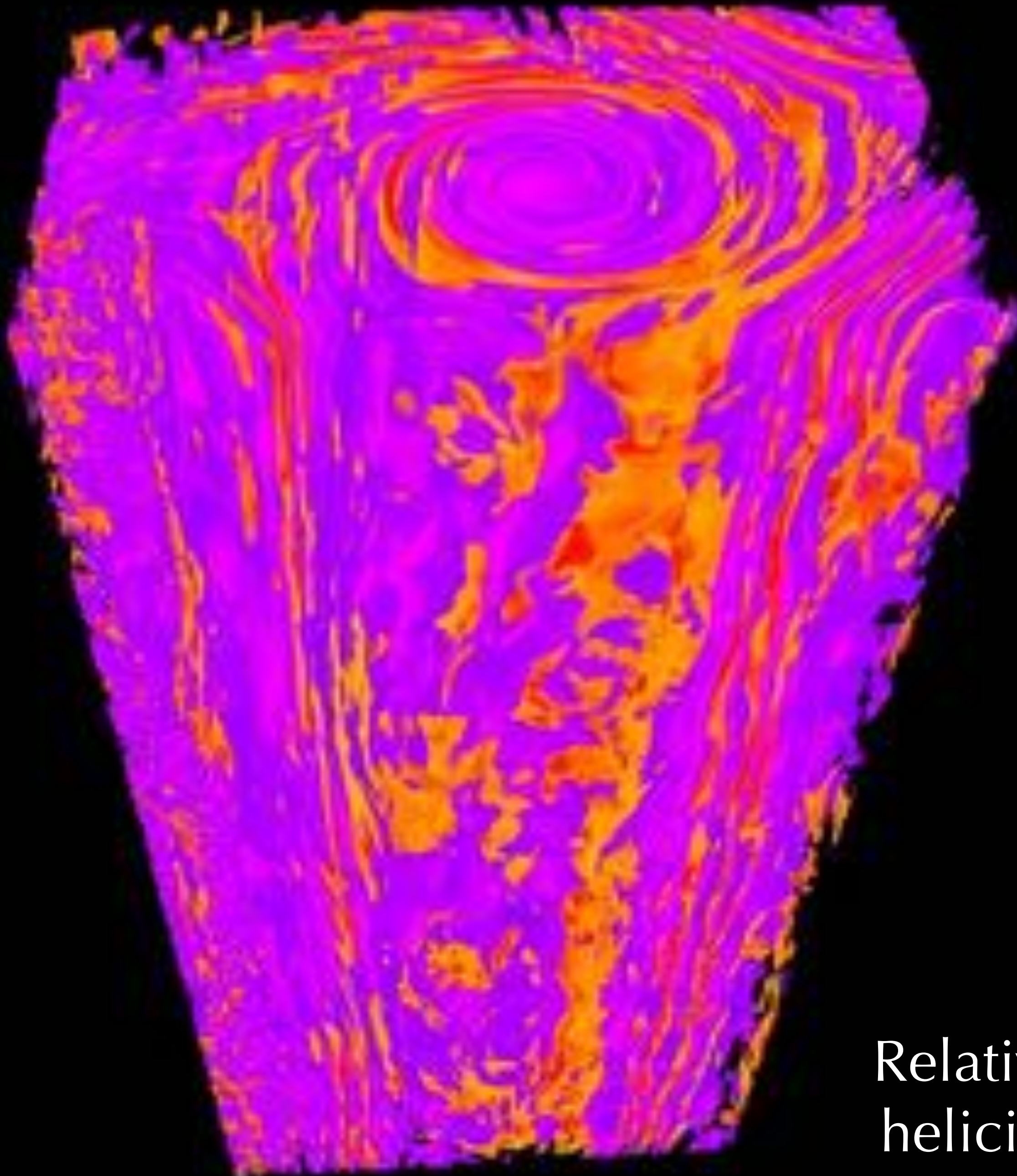


Mininni & Pouquet, PRE **79**, 026304 (2009), Phys. Fluids **22**, 035105 (2010), JFM **699**, 263 (2012)

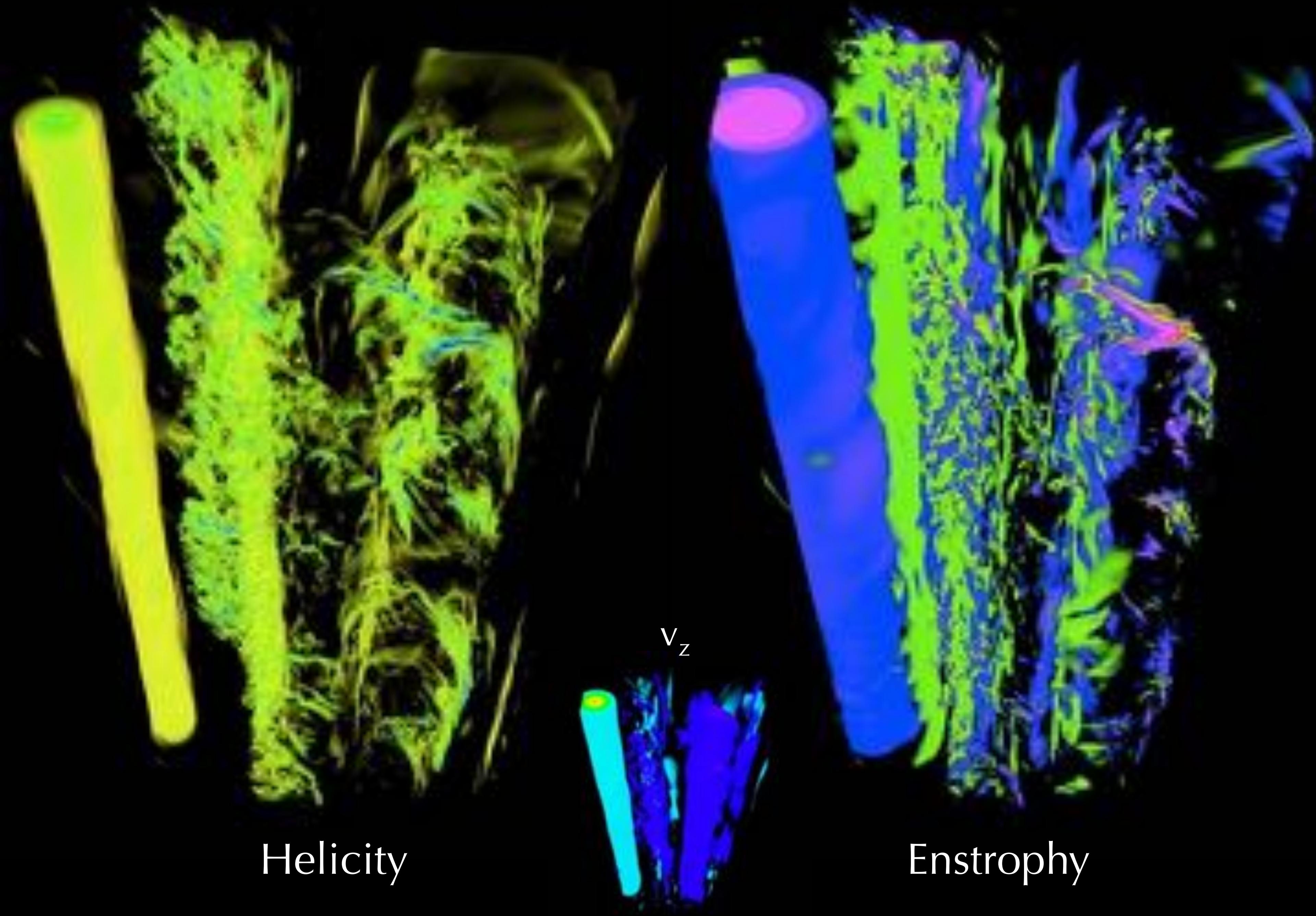


Enstrophy 1536^3


Helicity
v_z

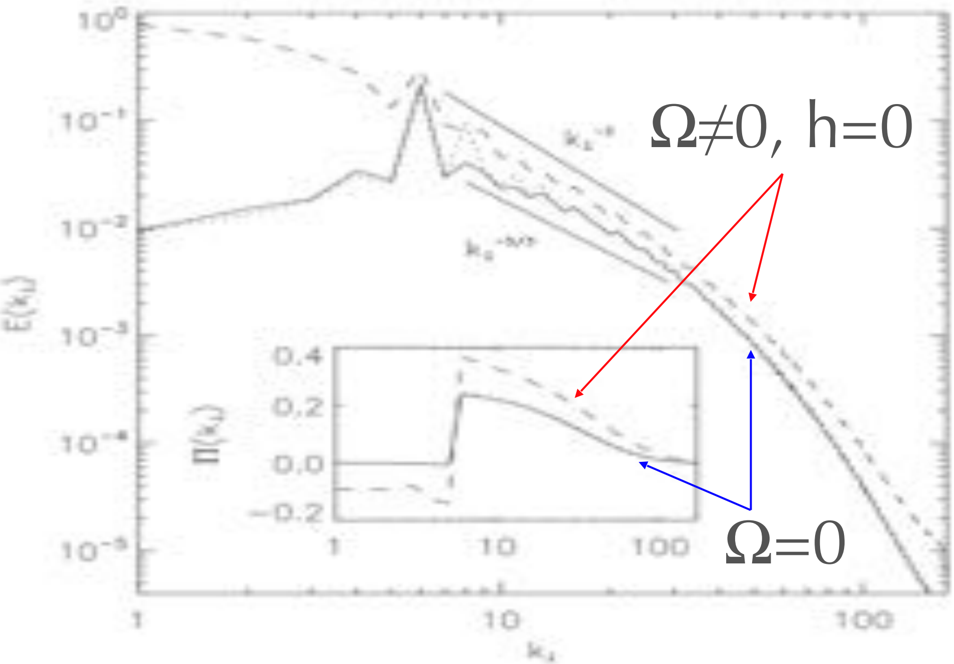


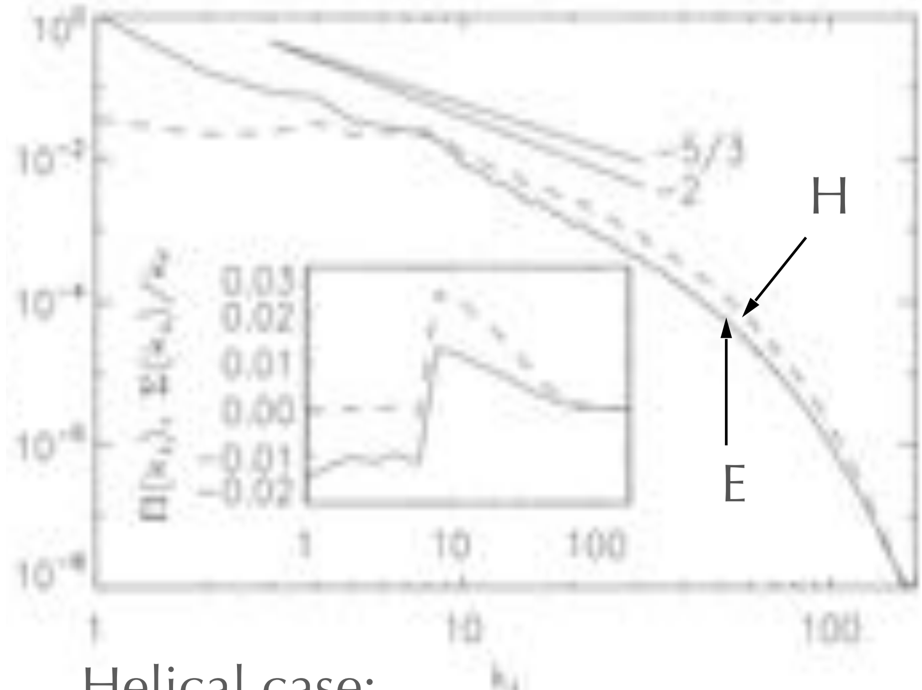
Relative
helicity



v_z
Helicity
Enstrophy

ENERGY SPECTRUM IN ROTATING FLOWS





Non-helical case:

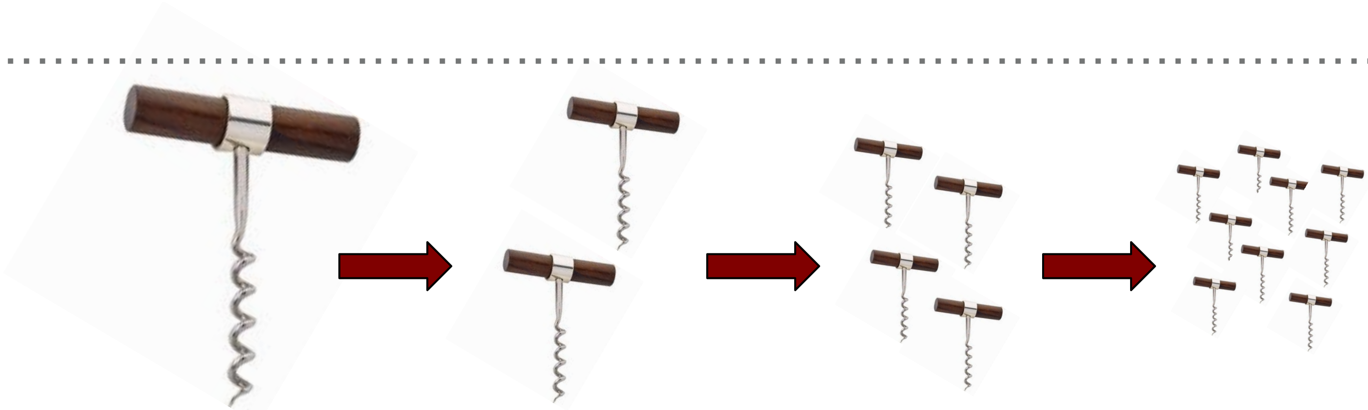
- An inverse cascade of energy develops for small Ro.
- The flow becomes anisotropic.
- The spectrum goes towards $k_\perp^{-2}$.

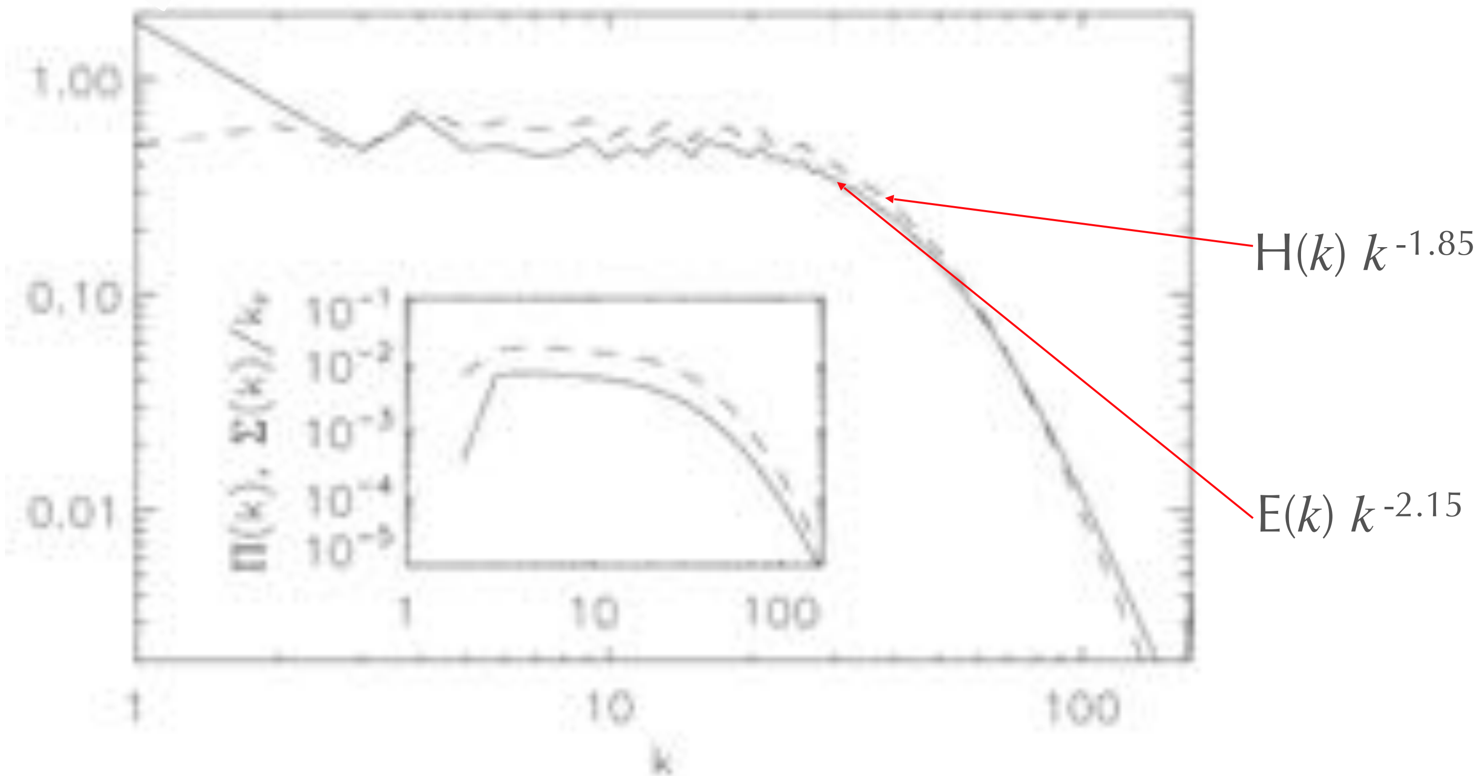
Helical case:

- Inverse cascade of energy and direct cascade of helicity.
- The direct energy flux is sub-dominant to the helicity flux.
- The energy spectrum becomes steeper than $k_\perp^{-2}$.

Mininni, Alexakis & Pouquet, PoF **21**, 015108; Mininni & Pouquet, PRE **79**, 026304 (2009)

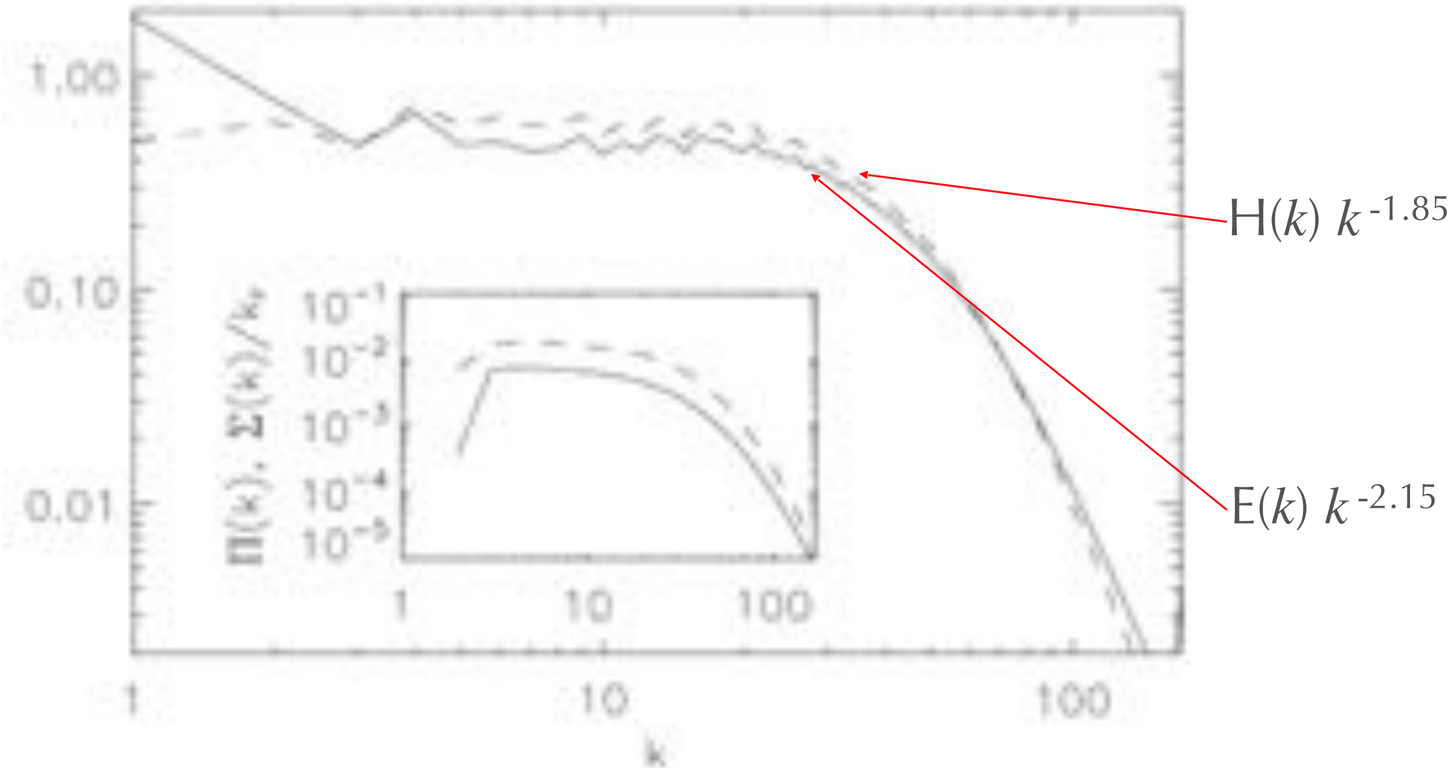
THE HELICITY CASCADE





Mininni & Pouquet, PRE **79**, 026304 (2009), Phys. Fluids **22**, 035105 (2010), JFM **699**, 263 (2012)

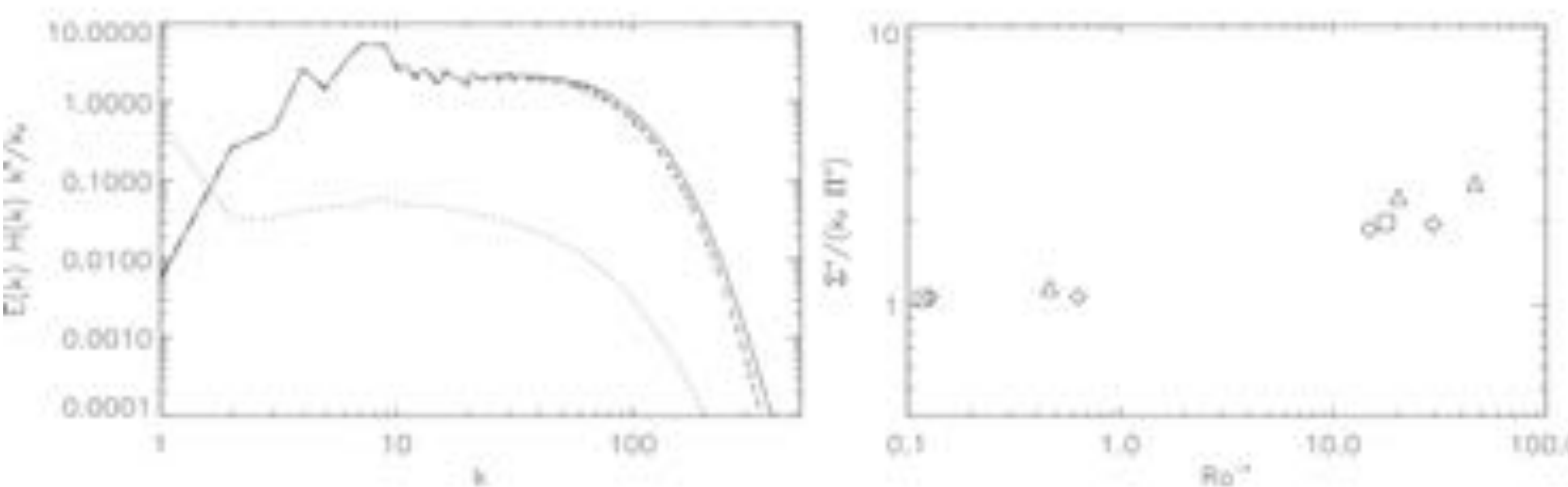
HELICAL ROTATING TURBULENCE



- With rotation, energy goes towards large scales and helicity dominates the direct cascade: the helicity flux is constant $\delta \sim h_l \tau_\Omega / \tau_l^2 \sim h_l u_l^2/(l_\perp^2\, \Omega)$, and $h_l \sim l_\perp^2/u_l^2$.
- If $E(k_\perp) \sim k_\perp^{-n}$, $H(k_\perp) \sim k_\perp^{-4+n}$ or $E(k_\perp)H(k_\perp) \sim k_\perp^{-4}$
- From Schwarz, $n \leq 2.5$ (the equality corresponds to maximum helicity).

Mininni & Pouquet, PRE **79**, 026304 (2009), Phys. Fluids **22**, 035105 (2010), JFM **699**, 263 (2012)

THE K^{-4} SPECTRUM AND THE DIRECT HELICITY FLUX



- The product of the energy and helicity spectra follow a $\sim k_\perp^{-4}$ law in several runs with rotation and helicity.
- The amount of helicity flux that goes towards small scales (normalized by the direct energy flux) increases with decreasing Rossby number, indicating the dominance of a direct cascade of helicity. Baerenzung et al., JAS (2011).
- The “n+m = 4” rule has been shown recently to be exact for rotating turbulence in the weak turbulence regime (Galtier 2014).

INTERMITTENCY: STRUCTURE FUNCTIONS

- For a component of a field *f* we define the longitudinal structure functions

$$\mathfrak{S}_p^f(l) \equiv \langle |\delta f|^p \rangle$$

where the longitudinal increment is

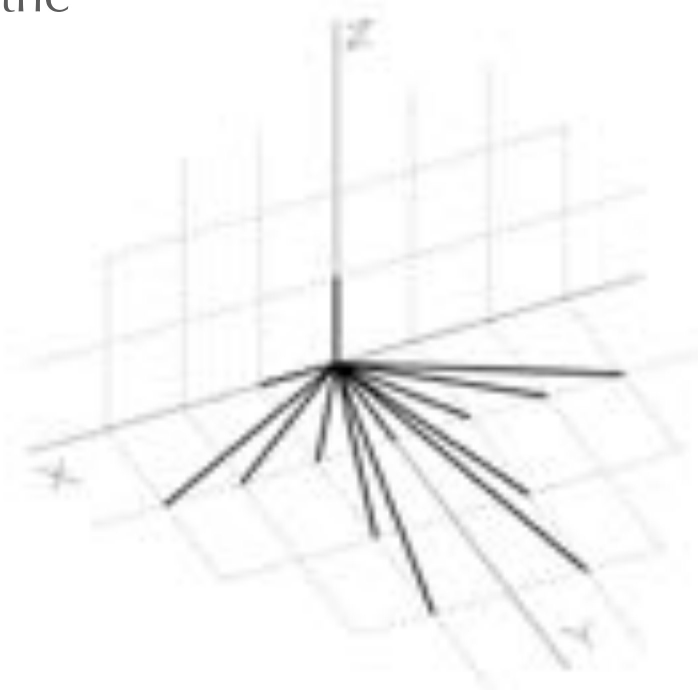
$$\delta \mathbf{f} = \mathbf{f}(\mathbf{x}+\mathbf{l}) - \mathbf{f}(\mathbf{x})$$

where **f** is in the direction of **l**.

- If the flow is self-similar we expect

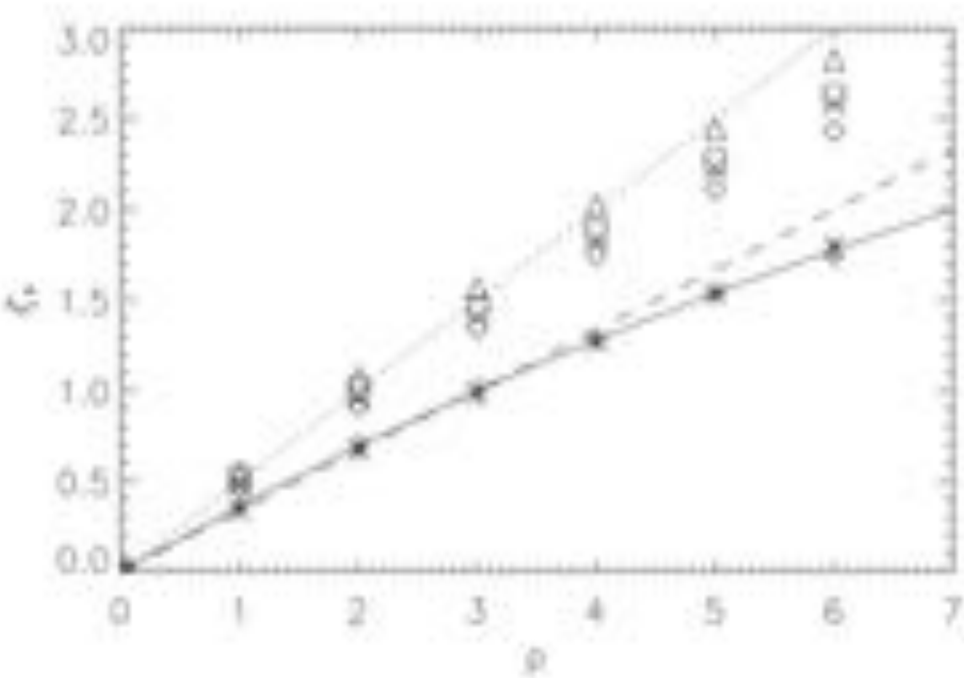
$$\mathfrak{S}_p^f(l) \sim l^{\zeta_p^f}$$

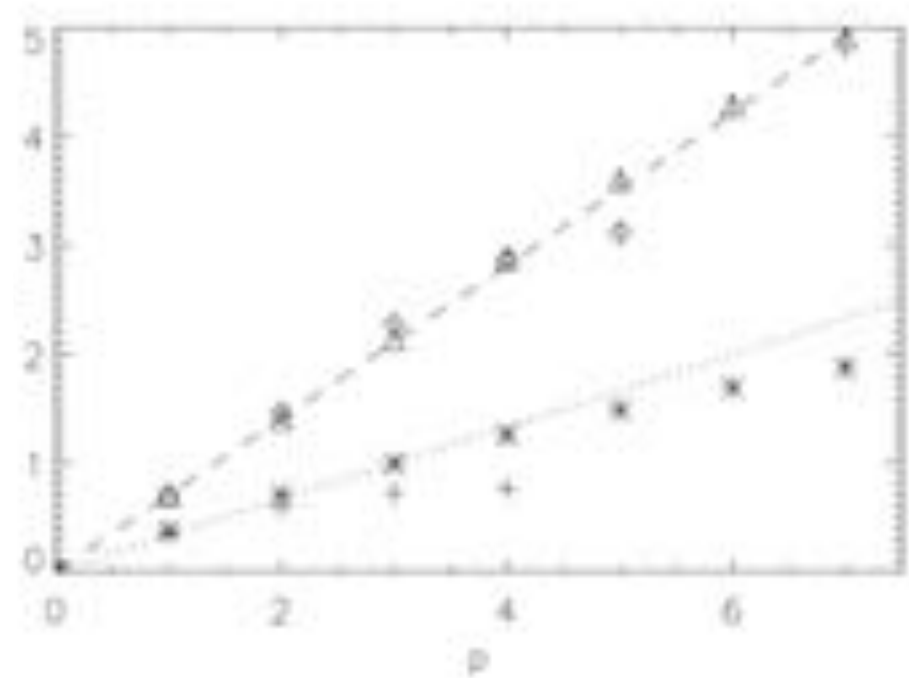
with the exponents linear in *p*.



- For isotropic turbulence then $\zeta_p = p/3$, for a non-helical rotating flow $\zeta_p = p/2$, and for the helical case $\zeta_p = 3p/4$.
- In practice departures from the straight line are observed, and the anomalous scaling observed in the data is the result of intermittency.

SCALING EXPONENTS





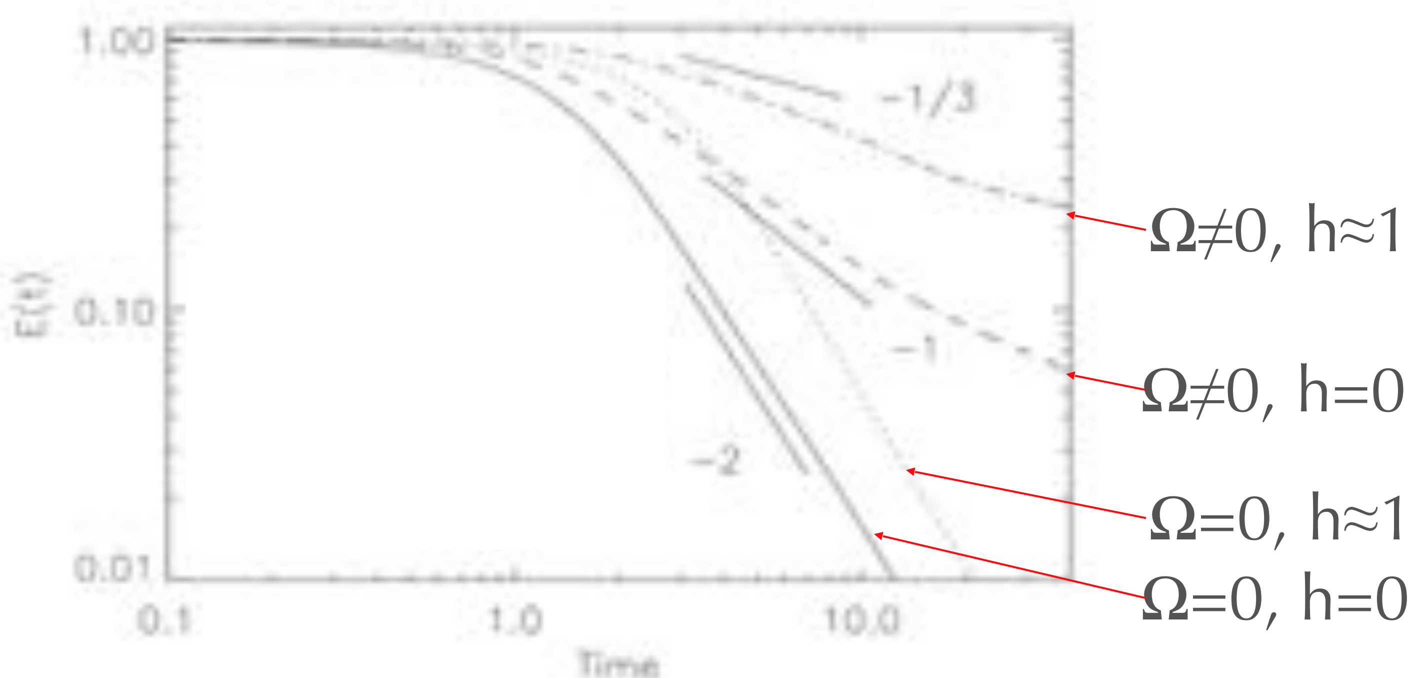
- Non-helical rotating turbulence is less intermittent than isotropic turbulence, but even at late times the exponents still deviate from a straight line.
- The second order exponent is close to the theoretical value of 1.
- Helical rotating turbulence is almost scale invariant.
- The second order exponent is close to 1.4 (for a flow with maximum helicity, 1.5 is predicted).

Mininni & Pouquet, Phys. Fluids **22**, 035105 (2010)

ARE THERE ANY IMPLICATIONS?

- Does the presence of helicity affect the decay of turbulence? Does it affect the lifetime of structures?
- Note different decay laws have been measured in simulations and experiments. Morize, Moisy, and Rabaud 2005; Morize and Moisy 2006, van Bokhoven et al. 2008, Davidson 2010.
- Does helicity affect the turbulent transport and diffusion of contaminants?

FREELY DECAYING FLOWS



- Simulations of bounded freely decaying turbulence, with and without rotation/ helicity.
- Without rotation, helicity plays no role in the decay, except for a delay of the beginning of the self-similar regime
- With rotation, the helical flow decays slower.
- The decay laws can be correctly predicted taking into account the presence of helicity.

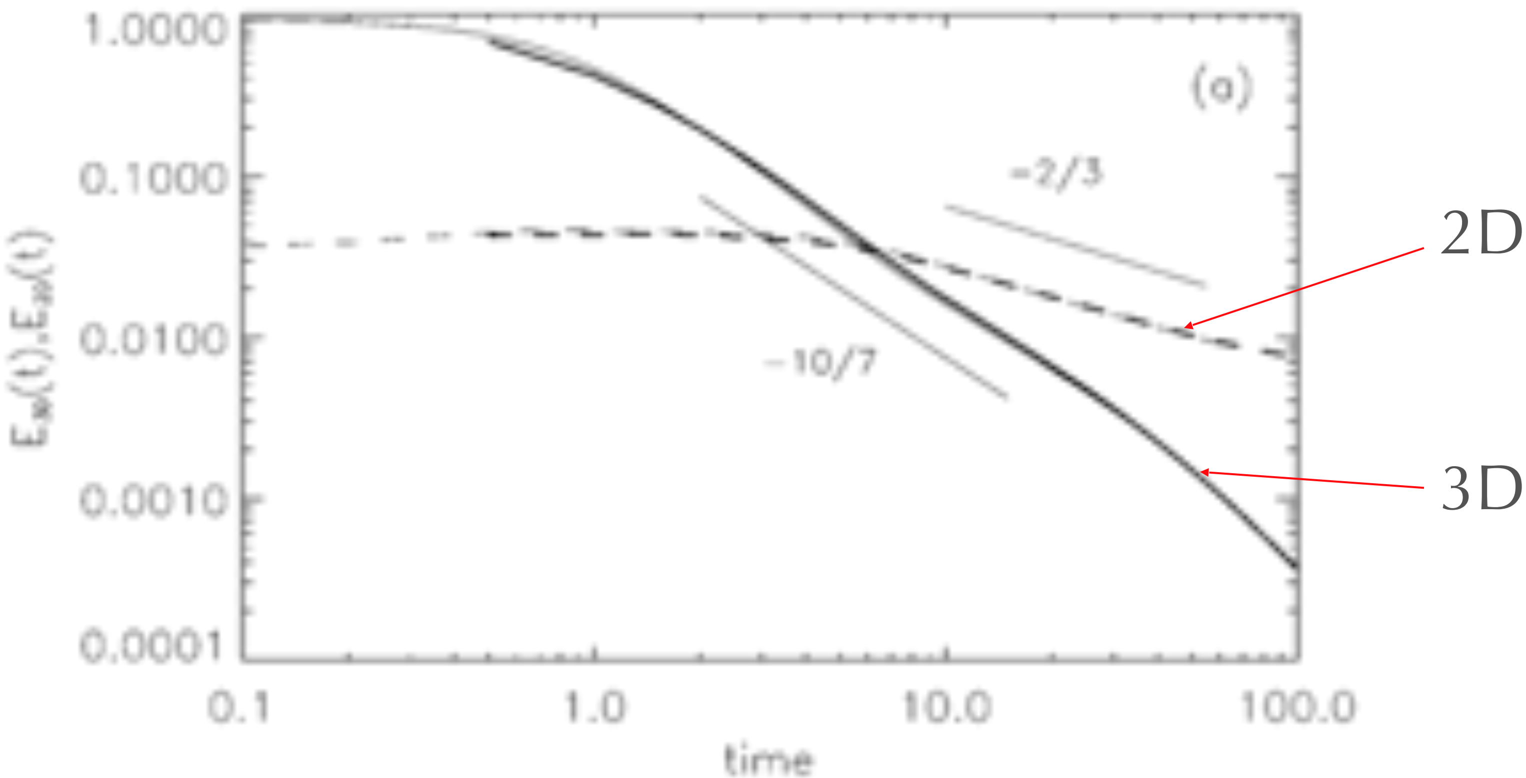
Teitelbaum & Mininni, PRL **103**, 014501 (2009)

"BOUNDED" FREELY DECAYING TURBULENCE

- From the energy balance: $$\frac{dE}{dt} \sim \epsilon$$
- In the absence of rotation: $$\frac{dE}{dt} \sim \frac{E^{3/2}}{L}$$
- If $L \sim L_0$ (constant), then $$E(t) \sim t^{-2}$$
- With rotation (no helicity): $$E(k) \sim \varepsilon^{1/2}\Omega^{1/2}k^{-2}$$ $$E(t) \sim kE(k)$$ $$\frac{dE}{dt} \sim \frac{1}{\Omega}\left(\frac{E}{L}\right)^2$$

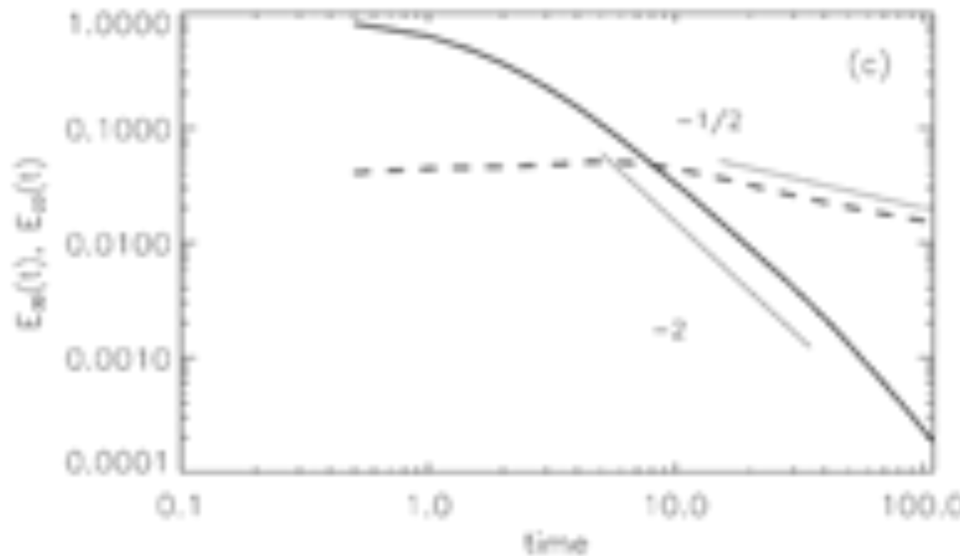
  which for constant L leads to $E(t) \sim t^{-1}$. Squires et al. 1994; Morize et al. 2005.
- Taking into account the helicity cascade, it leads to $E(t) \sim t^{-1/3}$.

SIMULATIONS OF DECAYING TURBULENCE

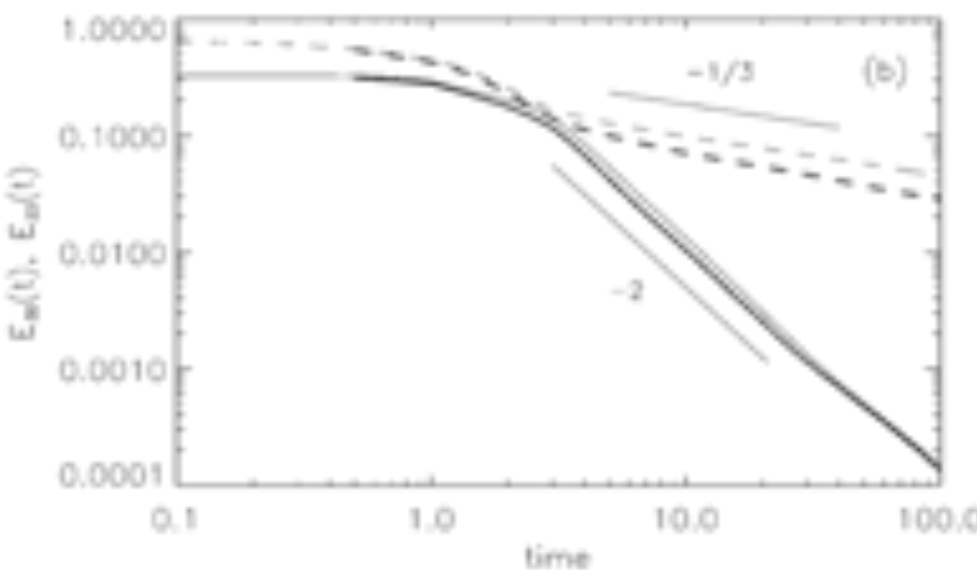


- Several DNS and LES with initial $E(k_\perp) \sim k_\perp^{\,3}$ large-scale spectrum.
- If the parallel integral scale has not saturated, the 3D modes decay as in non-rotating turbulence!
- Is that all?

A VARIETY OF DECAY LAWS

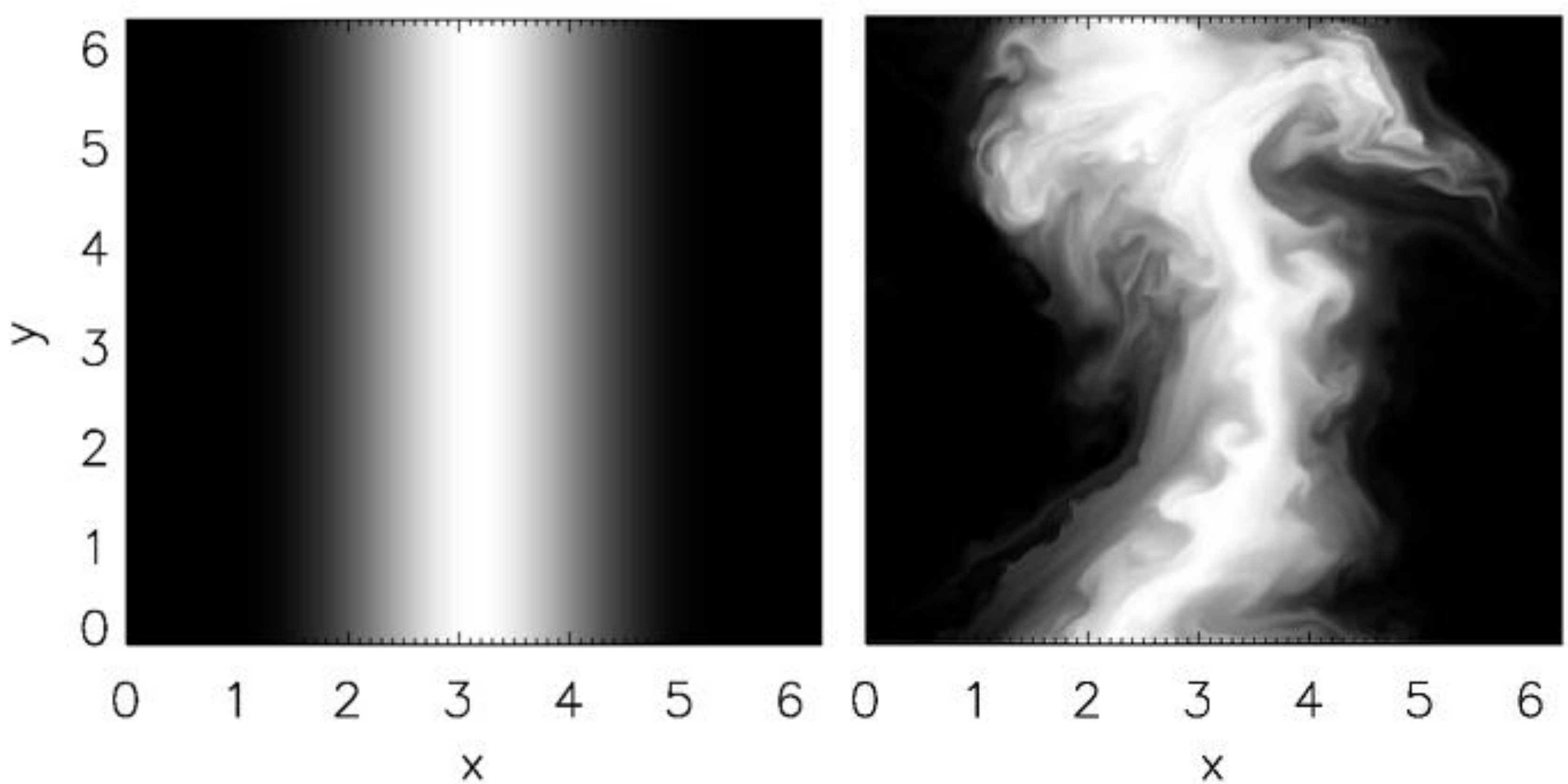


- "Unbounded" 2D modes (with $\sim k_\perp$ large-scale energy spectrum, with "bounded" 3D decay.



- "Bounded" 2D helical decay with "bounded" 3D helical decay.

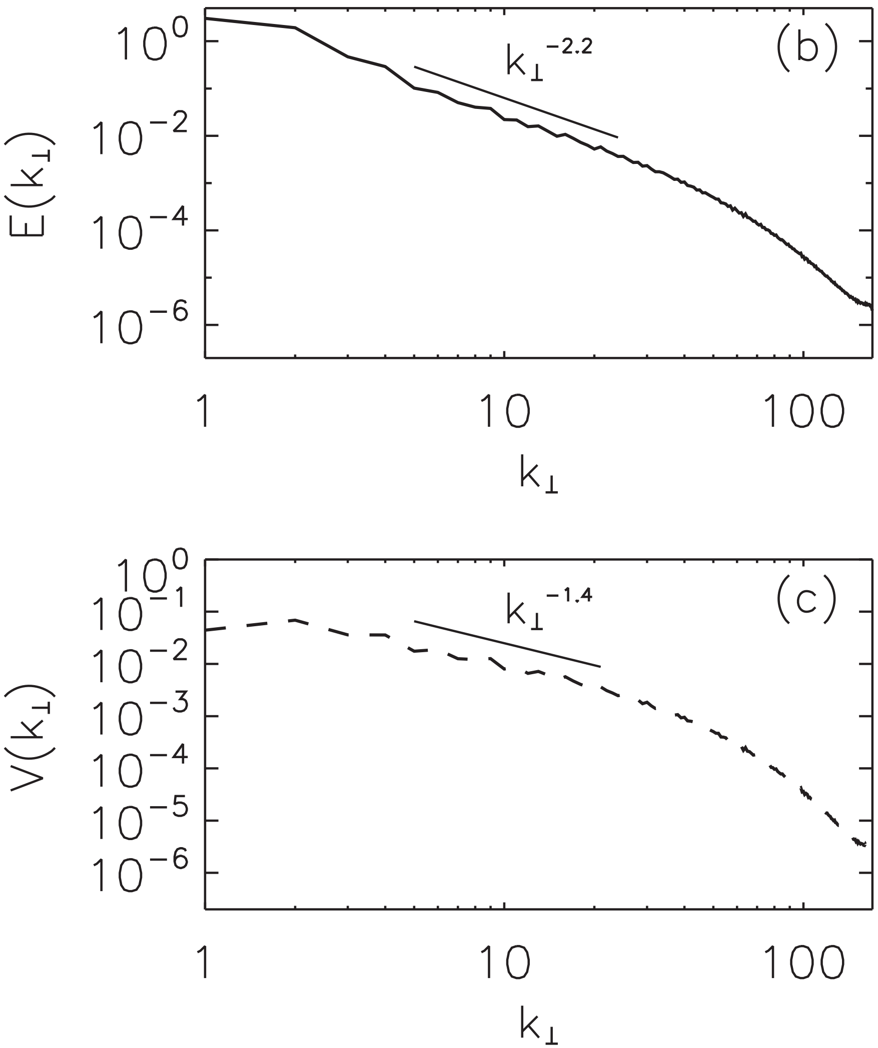
TRANSPORT AND MIXING

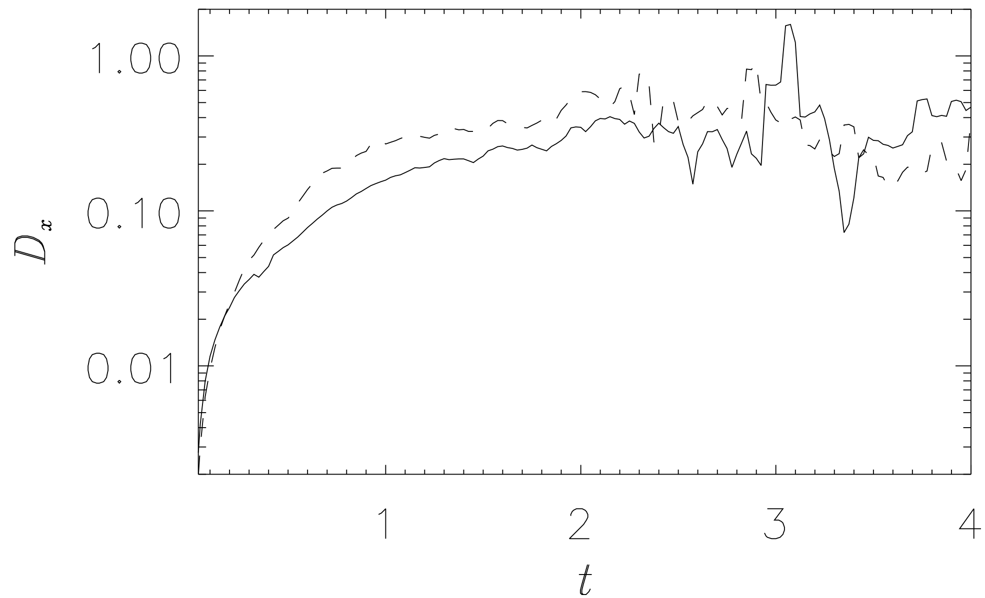


- Horizontal turbulent diffusion of a passive scalar is smaller in rotating helical flows than in rotating non-helical flows.

Rodriguez Imazio & Mininni, PRE **87**, 023018 (2013)

TRANSPORT AND MIXING





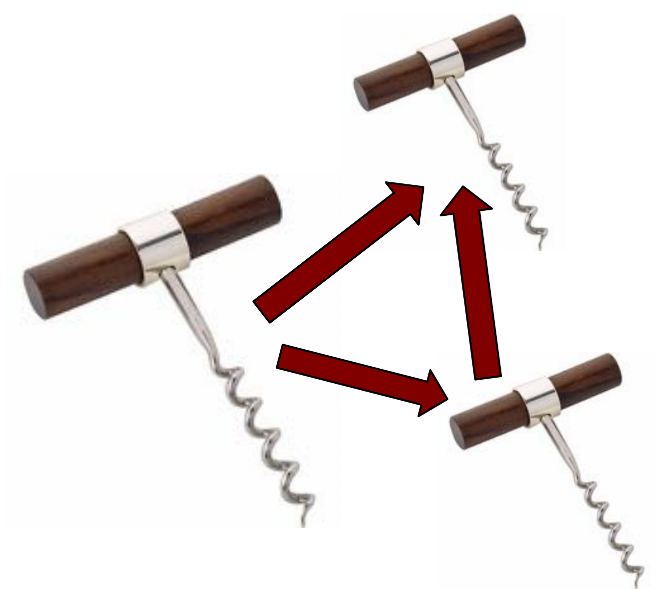
Rodriguez Imazio & Mininni, Phys. Scripta (2013)

REGULARITY

- From the momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{v} + \mathbf{F}$$

$$\Rightarrow \frac{dE(k)}{dt} = -\sum_{p,q} \int \mathbf{v}_k \cdot \left[\left(\mathbf{v}_p \cdot \nabla \right) \mathbf{v}_q \right] d^3x - 2\nu Z(k) + \varepsilon(k)$$

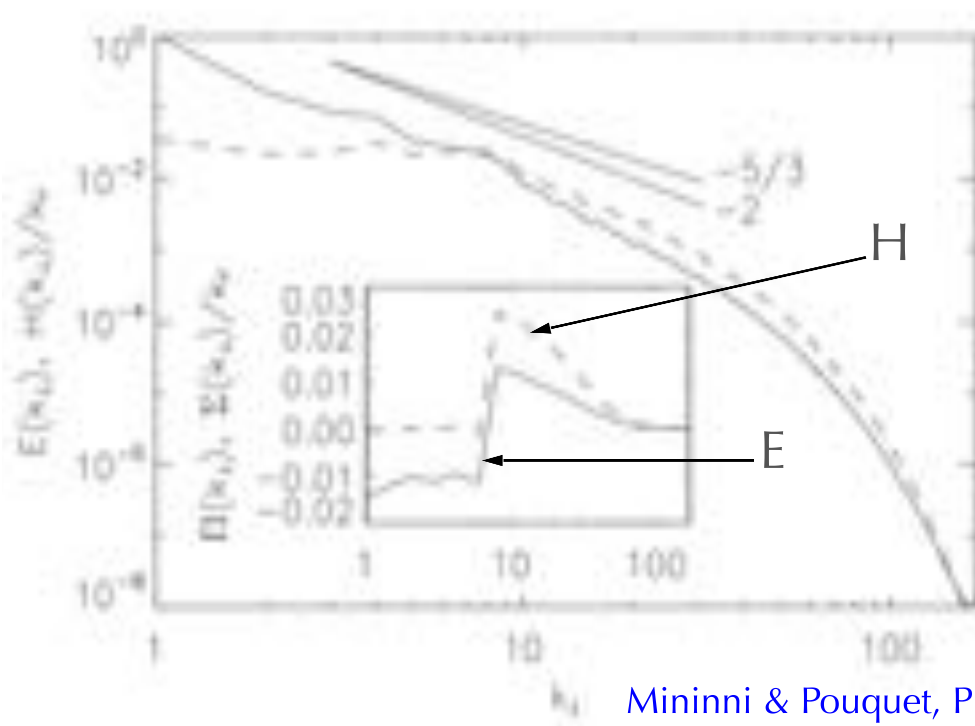


Biferale & Titi (2013)

REGULARITY

- A helical-decimated version of 3D Navier-Stokes displays an inverse cascade of energy, with a direct cascade of helicity.
- The system also has regular solutions (i.e., no singularity).

Biferale & Titi (2013)



Mininni & Pouquet, PRE **79**, 026304 (2009)

CONCLUSIONS

- Rotating flows provide a relatively *simple* example to understand the effect of restitutive forces and of waves in turbulence, with applications at the large scales of many geophysical and astrophysical flows.
- The presence of inertial waves results in the dominance (at least at some scales) of resonant and near-resonant interactions, which lead the flow towards a quasi-2D state.
- The role of near-resonant interactions and eddies (in particular, the effect of sweeping at intermediate scales) cannot be trivially neglected.
- The accumulation of energy in 2D modes can drive an inverse cascade of energy towards large scales.
- Helicity can be a major player in this problem, affecting the energy scaling and the cascades.
- This has implications in intermittency, the decay of turbulence, transport and mixing, and regularity of solutions.