Wave Turbulence: foundations and challenges

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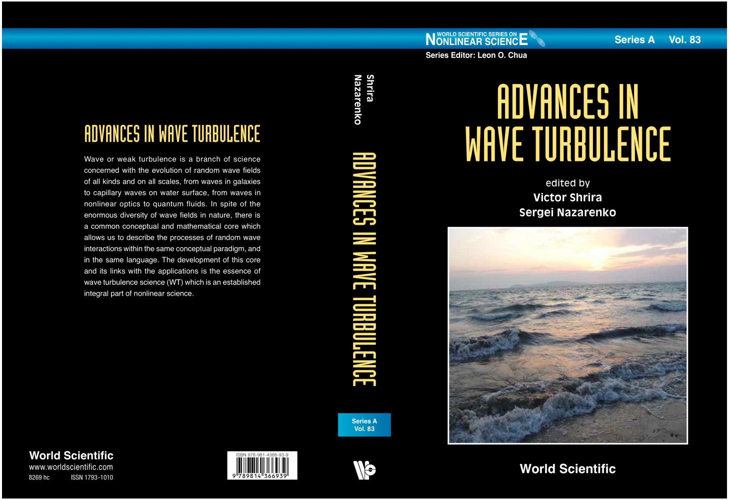
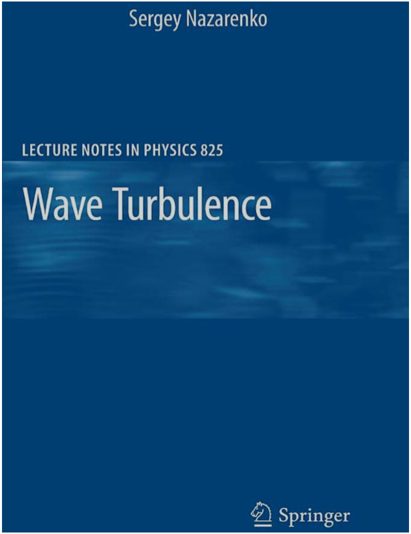
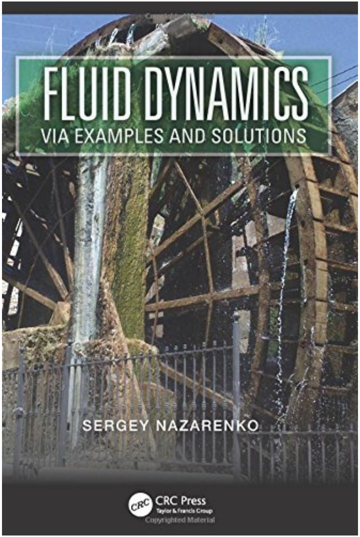
Book with focus on experiment



Book with focus on theory and new applications



Foundations of fluid dynamics, inc. waves and turbulence



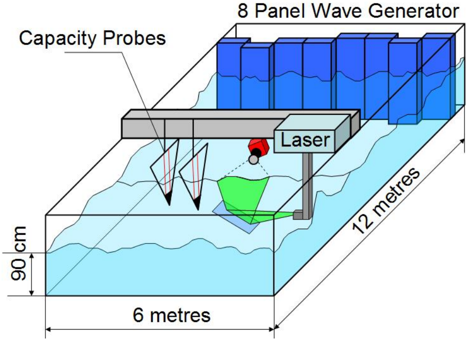
What is Wave Turbulence?

Wave Turbulence is a non-equilibrium statistical system of many randomly interacting waves. Kinetic equations of Wave Turbulence describe evolution of the wave energy in Fourier space.

Wave Turbulence on ocean surface

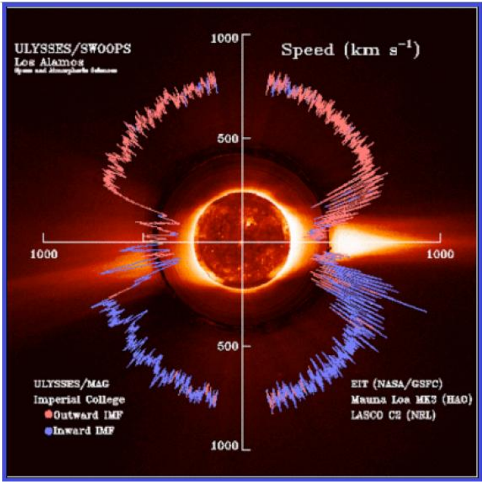
Kinetic equations of Wave Turbulence are used for the wave weather forecasts e.g. at ECMWF.

Waves in laboratory flumes



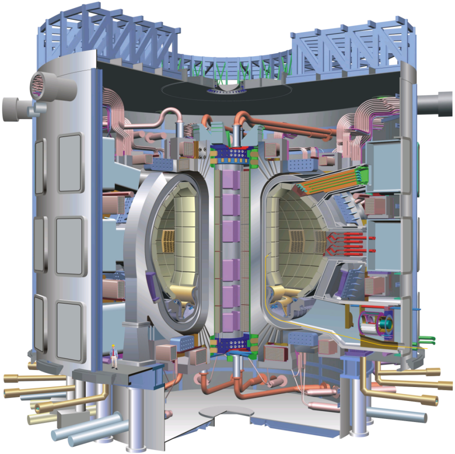
Gravity and capillary waves are the most actively studied in laboratory for validating the Wave Turbulence predictions.

Alfvén Wave Turbulence in solar wind



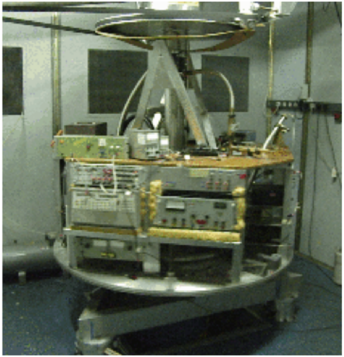
Alfvén Wave Turbulence is at the heart of MHD turbulence theory.

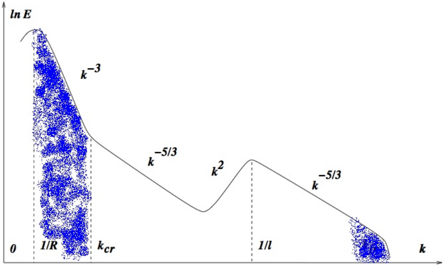
Waves in fusion plasmas



Drift Wave Turbulence is the cause of the anomalous energy and particle losses. Its interaction with zonal flows leads to Low-to-High transitions.

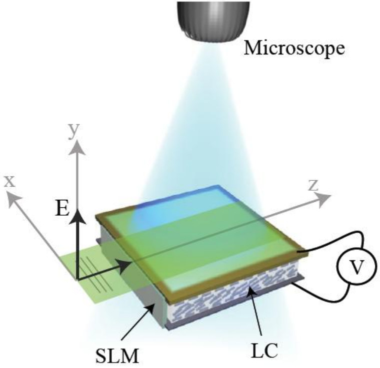
Superfluid Turbulence





Kolmogorov cascade is taken over by Kelvin Wave Turbulence at the classical-quantum crossover scale. Bottleneck effect.

Optical Wave Turbulence

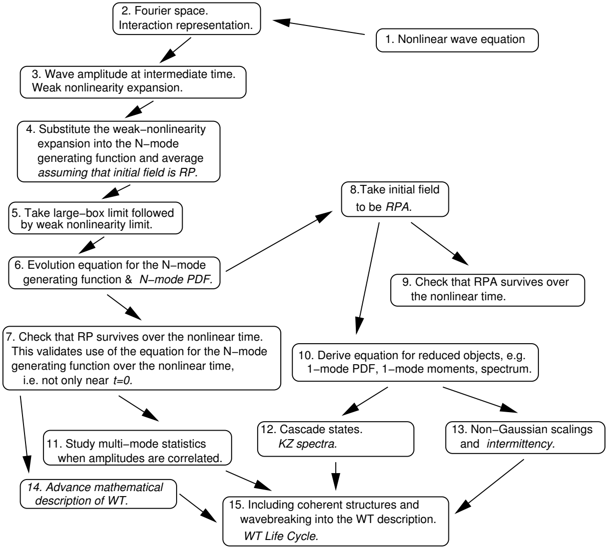


Optical Wave Turbulence: interacting waves and solitons

Challenges

- WT assumes small nonlinearity, which may not be uniformly valid across scales. How can we prove that what we see in experiment is WT? What is the role of interaction with strongly turbulent scales, coherent structures?
- Does the phase-amplitude randomness survive over nonlinear time?
- Controlling dissipation in experiment.
- To validate steps of the WT theory, one has to deal with joint PDF of wave modes. Tricky infinite-box limit.
- Unclear relation between the solutions of the kinetic equations, like Kolmogorov-Zakharov (KZ), and solutions for joint PDF.
- Non-stationary solution of the kinetic equations.

Wave Turbulence maze



Master example: Petviashvilli equation

Petviashvilli equation descrives drift waves in inhomoheneous plasmas in the limit $k\rho_s \ll 1$ in presence of scalar (thermal) nonlinearity:

$$\partial_t \psi = \nabla^2 \partial_x \psi - \psi \partial_x \psi \ , \tag{1}$$

where $\psi = \psi(x, y, t)$ is a real function of two space coordinates (x, y) and time t.

Petviashvilli equation conserves "energy":

$$E = \frac{1}{2} \int \psi^2 \, d\mathbf{x} = \text{const},$$

and "potential enstrophy":

$$\Omega = \frac{1}{2} \int \left[(\nabla \psi)^2 + \frac{1}{3} \psi^3 \right] \, d\mathbf{x} = \text{const}.$$

Note: Hamiltonian is actually given by Ω, not E:

$$\partial_t \psi = -\partial_x \frac{\delta \Omega}{\delta \psi}.$$

Fourier space

Let us consider a double-periodic system, $\mathbf{x} \in \mathbb{T}^2$, with period L in both directions Using Fourier coefficients

$$a_{\mathbf{k}}(t) = \frac{1}{L^2} \int_{\mathrm{Box}} \psi(\mathbf{x}.t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \ , \qquad (2)$$

rewrite Petviashvilli equation as

$$\dot{a}_k + ik_x k^2 a_k + i \sum_{1,2} a_1 a_2 k_{2x} \delta^k_{12} = 0, \qquad (3)$$

where dot is for time derivative, $a_{1,2} \equiv a_{\mathbf{k}_{1,2}}$, and $\delta^k_{12} = \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)$. In the linear limit,

$$a_k = A_k e^{-i\omega_k t} \ , \qquad (4)$$

where $A_k \in \mathbb{C}$ is a time-independent amplitude of the wave with wavevector $\mathbf{k}$, and the frequency of this wave is

$$\omega_k = k_x k^2 \ . \qquad (5)$$

Interaction Representation

Let us separate the time scales by introducing interaction representation variables as

$$b_k = \frac{a_k\, e^{-i\omega_k t}}{\epsilon |k_x|^{1/2}} \ , \qquad (6)$$

where we introduced parameter $\epsilon > 0$ for easier nonlinearity power counting.

We now have:

$$i\dot{b}_k = \epsilon \operatorname{sign}(k_x) \sum_{1,2} V_{12k} b_1 b_2 \delta^k_{12} e^{i\omega^k_{12} t} \ , \qquad (7)$$

where we have introduced the interaction coefficient

$$V_{12k} = \frac{1}{2}\sqrt{|k_x k_{1x} k_{2x}|} \ . \qquad (8)$$

and $\omega^k_{12} = \omega_k - \omega_1 - \omega_2$. Note that the linear term $\omega_k a_k$ is gone, and that there is explicit time-dependence in the nonlinear term. We have not made any approximations yet.

Discrete $\mathbf{k}$-space

Since we consider a periodic system, $\mathbf{x} \in \mathbb{T}^2$, the wavenumbers are discrete, $\mathbf{k} \in \mathbb{Z}^2$ (hereafter for simplicity $L = 2\pi$). Let the total number of modes be finite and bounded by some k_{max} (eg. a dissipation cutoff at high wavenumbers). Denote by B_N the set of all wavenumbers $\mathbf{k}_l$ inside the $\mathbf{k}$-space box of volume $(2k_{\text{max}})^2$:

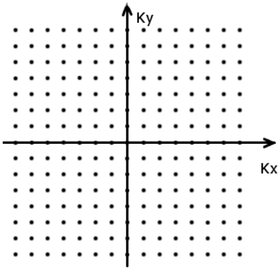


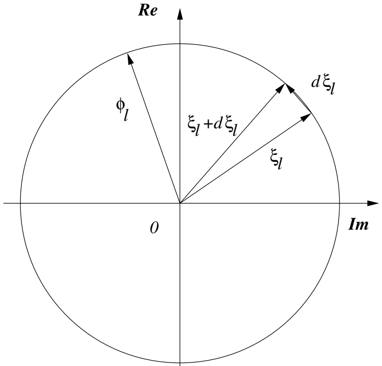
Figure: Set of active wave modes, $B_N \subset \mathbb{Z}^2$.

Amplitude-Phase representation

We write the wave function in terms of its amplitude and phase

$$a(\mathbf{k},t) = \sqrt{J_{\mathbf{k}}}\phi_{\mathbf{k}} \ ,$$

where $J_{\mathbf{k}} \in \mathbb{R}^+$ is the intensity and $\phi_{\mathbf{k}} \in \mathbb{S}^1$ is the phase factor of the mode $\mathbf{k}$. By $\mathbb{S}^1$ we mean the unit circle in the complex plane, i.e. $\phi_{\mathbf{k}} = e^{i\varphi_{\mathbf{k}}}$:



Probability Density Functions.

Denote the set of all $J_{\mathbf{k}}$ and $\phi_{\mathbf{k}}$ with $\mathbf{k}$ such that $\mathbf{k} \in B_N$ as $\{J, \phi\}$.
The probability of finding $J_{\mathbf{k}}$ inside $(s_{\mathbf{k}}, s_{\mathbf{k}} + ds_{\mathbf{k}}) \subset \mathbb{R}^+$ and finding $\phi_{\mathbf{k}}$ on the arch $(\xi_{\mathbf{k}}, \xi_{\mathbf{k}} + d\xi_{\mathbf{k}}) \subset \mathbb{S}^1$ (see Figure) is given in terms of the joint PDF $\mathcal{P}^{(N)}\{s, \xi\}$ as

$$\mathcal{P}^{(N)}\{s, \xi\} \prod_{\mathbf{k} \in B_N} ds_{\mathbf{k}} |d\xi_{\mathbf{k}}| \ . \tag{9}$$

M-mode joint PDF ($M < N$):

$$\mathcal{P}^{(M)}_{\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_M} = \left(\prod_{\mathbf{k} \neq \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_M} \int_0^\infty ds_{\mathbf{k}} \oint_{\mathbb{S}^1} |d\xi_{\mathbf{k}}| \right) \mathcal{P}^{(N)}\{s, \xi\} \ . \tag{10}$$

N-mode amplitude-only PDF obtained by integrating out all the phases,

$$\mathcal{P}^{(N,a)}\{s\} = \left(\prod_{\mathbf{k} \in B_N} \oint_{\mathbb{S}^1} |d\xi_{\mathbf{k}}| \right) \mathcal{P}^{(N)}\{s, \xi\} \ . \tag{11}$$

RP fields.

Single-mode amplitude PDF, $\mathcal{P}^{(1,a)} \equiv \mathcal{P}^{(a)}_{\mathbf{k}}(s_{\mathbf{k}})$, is obtained via integrated out all phases ξ, and all amplitudes s but one, $s_{\mathbf{k}}$.
Definition. Random phase (RP) field: all ϕ are independent random variables (i.r.v.) each uniformly distributed on $\mathbb{S}^1$.
Thus for a RP field

$$\mathcal{P}^{(N)}\{s,\xi\} = \frac{1}{(2\pi)^N}\mathcal{P}^{(N,a)}\{s\} \ .$$

Note: RP (in addition to the weak nonlinearity) is enough for the lowest level WT closure leading to an equation for the N-point amplitude-only PDF. However, it is not sufficient for the one-point WT closure, in particular the wave kinetic equation, and we need to assume something about the amplitudes too.

Random Phase and Amplitude (RPA) field definition

1. All the amplitudes and all the phases are i.r.v.,
2. All the phases are uniformly distributed on $\mathbb{S}^1$,
3. For RPA fields, the PDF has a product-factorized form,

$$\mathcal{P}^{(N)}\{s, \xi\} = \frac{1}{(2\pi)^N} \prod_{\mathbf{k}_l \in B_N} \mathcal{P}_j^{(a)}(s_j). \tag{12}$$

We have changed the standard meaning of RPA which usually stands for "Random phase approximation". In our definition of RPA:

1. The amplitudes are random, not only the phases.
2. RPA is defined as a property of the field, not an approximation.

RPA does not mean Gaussianity because it does not specify $\mathcal{P}_j^{(a)}(s_j)$. For Gaussian fields $\mathcal{P}^{(a)}(s_j) = \frac{1}{\langle J_j \rangle} \exp\left[-\frac{s_j}{\langle J_j \rangle}\right]$. WT does not require Gaussianity, only RPA, so we can study non-Gaussian intermittent fields!

Wave spectrum

The *wave spectrum* is defined as follows

$$n_k = \langle J_k \rangle \, .$$

For the infinite-box limit,

$$\langle \psi_k, \psi^*_{k'} \rangle = n_k \delta(\mathbf{k} - \mathbf{k}') \ ,$$

where $\delta(\mathbf{x})$ is the Dirac's delta function.
In terms of the generating function and the PDF, the wave spectrum can be expressed as follows,

$$n_k = \int_0^\infty s_k \mathcal{P}^{(a)}(s_k) ds_k \ .$$

Assumptions in the wave turbulence theory

- Weak nonlinearity.
- Initial RP statistics.

Equation for the PDF.

We have the following equation for the PDF,

$$\dot{\mathcal{P}} = 8\pi \int_{k_{jx},k_{mx},k_{nx}>0} |V_{mnj}|^2 \delta(\omega^j_{mn}) \delta^j_{m+n} \left[\frac{\delta}{\delta s}\right]_3 \left(s_j s_m s_n \left[\frac{\delta}{\delta s}\right]_3 \mathcal{P} \right) d\mathbf{k}_j d\mathbf{k}_m d\mathbf{k}_n,$$

$$\left[\frac{\delta}{\delta s}\right]_3 = \frac{\delta}{\delta s_j} - \frac{\delta}{\delta s_m} - \frac{\delta}{\delta s_n}.$$

No phases: phase randomness propagated. Amplitudes not separated: amplitude randomness only in coarse-grained sense.

Multiplying by s_k and integration over all s_j, we get the kinetic equation:

$$\dot{n}_k = 4\pi \int n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}}\right] \times \delta(\omega_k + \omega_{k_3} - \omega_{k_1} - \omega_{k_2}) \delta(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3.$$

Hydrodynamic turbulence, Richardson cascade

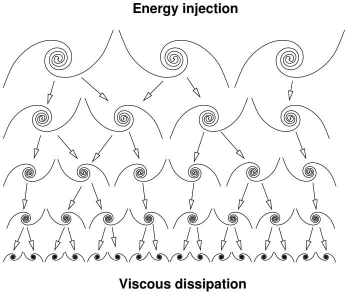


Figure: Richardson cascade in the physical space

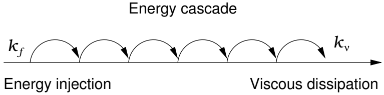


Figure: Richardson cascade in the *k*-space space

Kolmogorov spectrum

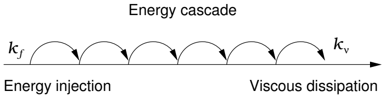


Figure: Richardson cascade in the k-space space

Richardson cascade states are characterized by Kolmogorov spectrum

$$E = CP^{2/3}k^{-5/3}.$$

WT cascade. Kolmogorov-Zakharov spectrum

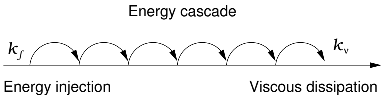


Figure: WT cascade in the k-space space

WT cascade states are characterized by Kolmogorov-Zakharov (KZ) spectrum

$$E = CP^{\sigma} k^{\nu}.$$

KZ spectrum can be anisotropic, e.g. for Petviashvili $E \sim k_x^{\nu_x} k_y^{\nu_y}$.

Solutions for the PDF

An arbitrary function of the un-averaged energy $E = \int \omega_k s_k \, d\mathbf{k}$ is is a steady solution,

$$\dot{\mathcal{P}}(E) = 0.$$

This property is common for all Liouville-type N-particle equations. An important special case is given by the exponential function,

$$\mathcal{P} = e^{-\beta \int \omega_k s_k \, d\mathbf{k}},$$

where β is an arbitrary constant. To understand the meaning of this solution, let us write its discrete version:

$$\mathcal{P} = \prod_j^N e^{-\beta \omega_j s_j}.$$

This solution describes a thermodynamic equilibrium, corresponding to N statistically independent Gaussian-distributed modes with a mean intensity given by a Rayleigh-Jeans spectrum, $n_k \sim 1/\omega_k$.

Solutions for the PDF

This kind of solutions were already discussed in the very first WT paper by Peierls. However, if we replace RJ spectrum with another stationary solution of the kinetic equation, the KZ spectrum, then it is easy to see that the result is not a solution of the PDF equation-not even an approximate solution! Thus the KZ solution obtained in the one-mode WT closure is valid only in some "coarse-grained" sense. However, it remains to be understood what this coarse-graining is in terms of the multi-mode solutions. It is also possible that for describing the KZ states in the multi-mode space one has to invoke amplitude fluxes.

Evolution of the one-mode PDF

Evolution equation for the one-mode PDF:

$$\frac{\partial P(t,s(k))}{\partial t} + \frac{\partial F}{\partial s(k)} = 0, \qquad (13)$$

with the probability flux

$$F = -s\left(\gamma P + \eta \frac{\partial P}{\partial s}\right) \qquad (14)$$

and

$$\begin{aligned}
\eta_k(t) &= 4\pi\epsilon^2 \int |W^{k1}_{23}|^2 \delta(\Delta k)\delta(\Delta\omega) n_1 n_2 n_3 dk_1 dk_2 dk_3, \\
\gamma_k(t) &= 8\pi\epsilon^2 \int |W^{k1}_{23}|^2 \delta(\Delta k)\delta(\Delta\omega)\Big[n_1(n_2+n_3) - n_2 n_3\Big] dk_1 dk_2 dk_3.
\end{aligned}$$

Gaussian fields with $P = \frac{1}{n_k} \exp\left[-\frac{s_k}{n_k}\right]$ satisfy the stationary equation (with $F=0$). But it is never a solution for evolving systems! What is?

Self-similar evolution of the spectrum. MHD turbulence.

non-stationary spectra preceding the KZ one can be described by self similar solutions

$$n(k,t) = \frac{1}{\tau^a} f(\eta)$$

with self-similar variable $\eta = k/\tau^b$, where $\tau = t^* - t$. Making the substitution into equation 1.1,

$$\frac{1}{\tau^{a+1}} \left(a f(\eta) + b\eta f'(\eta) \right) = \frac{1}{8} \iint_A \frac{\tau^{4b}}{\tau^{2a}} W_{\eta 12} f(\eta_1) \left[f(\eta_2) - f(\eta) \right] \mathrm{d}\eta_1 \mathrm{d}\eta_2,$$

where

$$W_{\eta 12} = \frac{\left(\eta_2^2 + \eta^2 - \eta_1^2\right)^2}{\eta_2 \eta_1 \eta^2} \sqrt{2\left(\eta_1^2\eta^2 + \eta_1^2\eta_2^2 + \eta^2\eta_2^2\right) - \eta_1^4 - \eta_2^4 - \eta^4}.$$

We want τ to be eliminated from both sides of the equation and so we get a relation on a and b, $a = 1 + 4b$ and our self-similar evolution equation

$$a f(\eta) + b\eta f'(\eta) = \frac{1}{8} \iint W_{\eta 12} f(\eta_1) \left[f(\eta_2) - f(\eta) \right] \mathrm{d}\eta_1 \mathrm{d}\eta_2. \qquad (2.1)$$

Self-similar evolution of the spectrum. MHD turbulence.

For the self-similar solution we have the following boundary conditions:

1. $f(\eta) \sim \eta^{-x}$ as $\eta \to 0$.
2. $f(\eta) > 0 \ \forall \ \eta$.
3. $f(\eta) \to 0$ as $\eta \to \infty$.

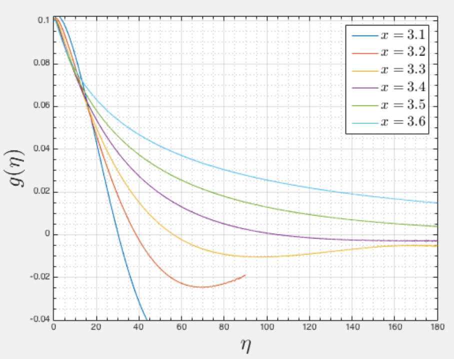
Find x for which the decay is fastest

The first boundary condition can be simplified by making the substitution $g(\eta) = f(\eta)\eta^x$ so that $g(0) \sim 1$. Making this substitution in equation 2.1 gives the equation which will be solved numerically;

$$\frac{\mathrm{d}g(\eta)}{\mathrm{d}\eta} = \frac{1}{8b} \iint\limits_{A} W_{\eta 12} \eta^{x-1} \eta_1^{-x} g(\eta_1) \left[\eta_2^{-x} g(\eta_2) - \eta^{-x} g(\eta)\right] \mathrm{d}\eta_1 \mathrm{d}\eta_2. \qquad (2.2)$$

We know also that $x = a/b$ and so along with the previous relation on a and b we have that $b = \frac{1}{x-4}$.

Self-similar evolution of MHD turbulence. Ongoing work with Nick Bell et al



Here $x^* = 2.82$. Simulations of the KE give $x^* = 2.33$ (Galtier et al 2000). Differential approximation of the first order gives $x^* = 2.33$; Differential approximation of the second order gives $x^* = 2.088$ (Thalabard et at 2015).

Discussion.

- Wave turbulence theory is an effective theory for describing turbulent states in a wide range of applications where random interacting waves play an important role. Examples: water waves, MHD waves, plasma waves, nonlinear optics, quantum fluids and BEC.
- WT is used for the operational sea wave forecast, for explaining turbulence in solar wind, for understanding LH transition in tokamaks.
- There remain challenges in implementing WT experimentally and in rigorous mathematical validation of the derivation steps.
- Need to extend WT on systems where random weak waves coexist with strongly nonlinear coherent structures, e.g. vortices and solitons.
- Incorporating WT as a part of description of complex systems, e.g. coarse-grained description of anomalous transport of energy, momentum and particles, subgrid modelling, etc.