Numerical modeling of atmospherically perturbed phase screens: new solutions for classical fast Fourier transform and Zernike methods

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We describe new solutions permitting us to overcome the well-known problems encountered when employing the two main classical methods for numerical modeling of atmospherically perturbed phase screens. The first method, the fast-Fourier-transform-based numerical method, suffers from a lack of low frequencies. Subharmonics adding is an already-known solution to this problem, but no criterion has been defined up to now in order to precisely determine how many subharmonics are necessary for each given case of physical and numerical characteristics. We define two criteria and show their practical efficiency. The second, Zernike-based, method suffers, *a contrario*, from bad behavior of the phase screens at high spatial frequencies. To overcome this problem, due to numerical instability, we developed an algorithm based on an alternative definition of the Zernike polynomials, involving the recurrence definition of the Jacobi polynomials, as well as the relationship between the Zernike and the Jacobi polynomials. The methods described and used in this paper have been implemented within the freely distributed software package CAOS.

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1. Introduction

Turbulence is mainly located within a few relatively narrow layers, at least in good astronomical sites. The finite number of turbulent layers is a consequence of a modelization of the profile of the structure constant of the refraction index fluctuations, $C_n^2(h)$, where h is the altitude, to an ensemble of discrete values. Each of these values corresponds to a turbulent layer of the atmosphere, and each turbulent layer can be physically simulated as a random phase screen, whose power spectrum follows a von Karman/Kolmogorov model.

The first step for building a turbulent atmosphere is, hence, to generate the phase screens that will simulate the behavior of each turbulent layer. Two methods are usually considered:

1. the fast-Fourier-transform (FFT)-based method, suffering, however, from a lack of low frequencies;
2. the Zernike-polynomials-based method, suffering *a contrario* from possible bad behavior of the phase screens at high spatial frequencies.

An already-known solution to the first problem consists of boosting the low-spatial-frequencies [1,2] by adding subharmonics to the FFT-computed phase screens. But the number of low frequencies to be added is not actually defined in the literature. Section 2 details the two criteria we have defined for this scope and shows their efficiency.

In Section 3 we show how the second problem can be overcome by considering an alternative definition of the Zernike polynomials for high orders.

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In fine, a brief summary and concluding remarks are given in Section 4.

2. Fast-Fourier-Transform-Based Method with Subharmonics Adding

The well-known FFT method allows us to generate phase screens $\varphi(\vec{r})$, where $\vec{r}$ is the two-dimensional position within the phase screen, assuming usually either a Kolmogorov or a von Karman spectrum $\Phi_\varphi(\vec{\nu})$, where $\vec{\nu}$ is the two-dimensional spatial frequency, from which is computed the modulus of $\tilde{\varphi}(\vec{\nu})$, the Fourier transform of $\varphi(\vec{r})$. Assuming the near-field approximation and small phase perturbation [3], the von Karman/Kolmogorov spectrum is given by

$$\Phi_\varphi(\vec{\nu}) = 0.0229 r_0^{-\frac{5}{3}} \left(\nu^2 + \frac{1}{\mathcal{L}_0^2}\right)^{-\frac{11}{6}}, \tag{1}$$

where r_0 is the Fried parameter and $\mathcal{L}_0$ is the wavefront outer scale (infinite for the Kolmogorov model). Within the framework of the classical FFT-based technique, a turbulent phase screen $\varphi_L(\vec{r})$ of physical length L is obtained by taking the inverse FFT of $\tilde{\varphi}_L(\vec{\nu})$, the modulus of which is obtained from Eq. (1) by applying the definition of the power spectrum, which is

$$\Phi_\varphi(\vec{\nu}) = \lim_{L\to\infty}\left(\frac{\langle|\tilde{\varphi}_L(\nu)|^2\rangle}{L^2}\right)$$
$$\Rightarrow |\tilde{\varphi}_L(\nu)| \simeq L r_0^{-\frac{5}{6}} \sqrt{0.0228}\left(\nu^2 + \frac{1}{\mathcal{L}_0^2}\right)^{-\frac{11}{12}}, \tag{2}$$

and which phase is random and uniformly distributed.

The obtained phase screen is thus numerically written

$$\varphi_L(i,j) = \sqrt{2}\sqrt{0.0228}\left(\frac{L}{r_0}\right)^{\frac{5}{6}} \left\{ \mathrm{FFT}^{-1}\left[\left(k^2 + l^2 + \left(\frac{L}{\mathcal{L}_0}\right)^2\right)^{-\frac{11}{12}} \exp\{\imath\theta(k,l)\}\right]\right\}, \tag{3}$$

where i and j are the indices in the direct space, k and l are the indices in the FFT space, {} stands for either *real part of* or *imaginary part of*, $\imath$ is the imaginary unit, and θ is the random uniformly distributed phase (between $-\pi$ and π). The factor $\sqrt{2}$ comes from the fact that here we use both the real and imaginary parts of the original complex generated FFT phase screens, which are independent one from the other [4]. This kind of phase screen suffers, however, from the lack of spatial frequencies lower than the inverse of the necessarily finite length L of the simulated array.

In order to compensate from this lack of low frequencies, the technique of subharmonics adding was proposed by Lane *et al.* [1]. This technique consists of generating additional low frequencies and adding their effects to the already-sampled frequencies. If one divides the lower unsampled spatial frequency content in three harmonics, thus adding the effect of $3 \times 3 = 9$ bidimensional spatial frequencies at each step, the subharmonics screens are numerically written in this case as

$$\varphi_{3^n L}(i,j) = \sqrt{2}\sqrt{0.0228}\left(\frac{L}{r_0}\right)^{\frac{5}{6}} \frac{1}{3^n} \left\{ \mathrm{DFT}^{-1}\left[\left(k_S^2 + l_S^2 + \left(\frac{L}{\mathcal{L}_0}\right)^2\right)^{-\frac{11}{12}} \exp\{\imath\theta(k_S,l_S)\}\right]\right\}, \tag{4}$$

where k_S and l_S are the indices in the discrete Fourier transform (DFT) space and n is the number of subharmonics added.

A tricky point remains of defining when to stop the iterative process of subharmonics adding. In their original paper, Lane *et al.* [1] add 5 subharmonics for a Kolmogorov screen over a pupil of 1 m × 1 m in order to match the theoretical structure function. Later on, Sedmak [2] adds 10 subharmonics to Kolmogorov screens and a number of subharmonics that gives a physical size comparable with, or a bit larger than, the outer scale of turbulence $\mathcal{L}_0$ in the case of von Karman screens. In a successive paper, Sedmak [5] also gives a thorough overview of the practical methods employable for adding subharmonics, with some refinement as well. Nevertheless, no real quantitative criterion for the number of subharmonics to be added has been defined in the literature, and we therefore treat this point hereafter.

A. Subharmonics-Adding Issue

We have investigated the question of how many subharmonics have to be added and defined two aid-to-decision criteria that are able to recommend a minimum number of subharmonics to be added in order to obtain a given precision.

Assuming that the number of computed phase screens is sufficiently large (ideally infinite), the decision criteria can be based on the computation of the ratio between theoretical relevant quantities and will-be-obtained quantities.

In the general case of von Karman turbulence, two candidates well fit the role of decision quantities: the integrated power over the whole range of frequencies and the structure function computed for a relevant space lag. In the case of an infinite wavefront outer scale (Kolmogorov model), the integrated power is infinite, so the only valid criterion becomes the structure function ratio.

1. *Integrated Power Ratio*

The total theoretical integrated power is obtained by integrating the power spectrum over the whole range of theoretically present frequencies, i.e., from 0 to infinity. This gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi_\varphi(\nu_x,\nu_y)\mathrm{d}\nu_x\mathrm{d}\nu_y = 2\pi\int_0^{\infty}\nu\Phi_\varphi(\nu)\mathrm{d}\nu$$
$$= 2\pi 0.0228 r_0^{-\frac{5}{3}} \times \int_0^{\infty}\nu\left(\nu^2+\frac{1}{\mathcal{L}_0^2}\right)^{-\frac{11}{6}}\mathrm{d}\nu$$
$$= \frac{6\pi}{5}0.0228\left(\frac{\mathcal{L}_0}{r_0}\right)^{\frac{5}{3}}. \quad (5)$$

While the will-be-obtained integrated power will not include frequencies lower than the lowest sampled one, which gives

$$\int_{\nu_{x,\mathrm{lim}}}^{\infty}\int_{\nu_{y,\mathrm{lim}}}^{\infty}\Phi_\varphi(\nu_x,\nu_y)\mathrm{d}\nu_x\mathrm{d}\nu_y = 2\pi 0.0228 r_0^{-\frac{5}{3}} \times \int_{\nu_{\mathrm{lim}}}^{\infty}\nu\left(\nu^2+\frac{1}{\mathcal{L}_0^2}\right)^{-\frac{11}{6}}\mathrm{d}\nu$$
$$= \frac{6\pi}{5}0.0228\left(\frac{\mathcal{L}_0}{r_0}\right)^{\frac{5}{3}} \times (1+\nu_{\mathrm{lim}}^2\mathcal{L}_0^2)^{-\frac{5}{6}}, \quad (6)$$

so that the integrated power ratio α_Φ obtained is simply

$$\alpha_\Phi = (1+\nu_{\mathrm{lim}}^2\mathcal{L}_0^2)^{-\frac{5}{6}}. \quad (7)$$

This already means that, because in the subharmonics-adding process $\nu_{\mathrm{lim}} = \frac{1}{3^n L}$, if the length of the screens L is chosen to be equal to the outer scale $\mathcal{L}_0$, this is not enough to guarantee that a sufficient number of low frequencies will be added. In fact, in that case $\alpha = 2^{-\frac{5}{6}} \simeq 0.561$, which is definitely smaller than 1.

Let us, hence, deduce a formula for the number of subharmonics to be added. From the last equation, one has

$$\alpha_\Phi^{-\frac{6}{5}} = 1+\left(\frac{\mathcal{L}_0}{3^n L}\right)^2 \Rightarrow n \geq \frac{1}{\ln 3}\ln\left[\frac{\mathcal{L}_0}{L}\left(\alpha_\Phi^{-\frac{6}{5}}-1\right)^{-\frac{1}{2}}\right], \quad (8)$$

from which we can immediately deduce that even by taking a phase screen length L equal to the outer scale $\mathcal{L}_0$, 1% accuracy is obtained only by adding at least two subharmonics (see Fig. 1).

2. *Structure Function Ratio*

Because the FFT-based phase screens are circularly periodic, the largest relevant structure function space lag ρ is equal to $L/2$. On the other hand, the definition of the structure function $D_\varphi(\vec{\rho})$ is

$$D_\varphi(\vec{\rho}) = 2\int\Phi_\varphi(\vec{\nu})[1-\cos(2\pi\vec{\nu}\vec{\rho})]\mathrm{d}\nu, \quad (9)$$

where the integral is made over the whole range of present frequencies, that means from 0 to infinity for the ideal case, and from ν_{lim} to infinity for the simulated case (assuming also an ideal frequency behavior).

In the Kolmogorov case ($\mathcal{L}_0$ infinite), the theoretical structure function is well known to be

$$D_\varphi(\vec{\rho}) = 6.88\left(\frac{\rho}{r_0}\right)^{\frac{5}{3}}, \quad (10)$$

while the will-be-obtained one is to be deduced from Eq. (9). The structure function ratio α_D is, thus, a function of n and L, the expression of which is

$$\alpha_D(n,L) = 0.0131 L^{-\frac{5}{3}}\int_{\frac{1}{3^n L}}^{\infty}\nu^{-\frac{8}{3}}[1-\cos(\pi L\nu)]\mathrm{d}\nu. \quad (11)$$

In the von Karman case, both structure functions are to be deduced from Eq. (9), and the resulting ratio α_D becomes a function of n and L, but also $\mathcal{L}_0$.

Figure 2 clearly shows the result of the computation of α_D for n ranging from 0 to 10, and L ranging

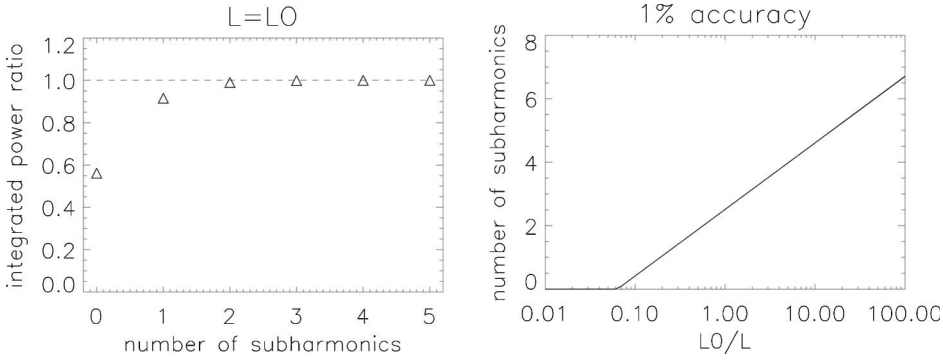

Fig. 1. Integrated power ratio α_Φ versus the number of subharmonics to be added n (left) and number of subharmonics to be added n versus the ratio $L/\mathcal{L}_0$ (right). The integrated power ratio α_Φ is computed for a screen length L equal to the wavefront outer scale $\mathcal{L}_0$. Note the values 0.561 for $n=0$ and 0.99 for $n=2$, reported in the text. The number of subharmonics to be added is computed for a requested accuracy of 1% ($\alpha_\Phi = 0.99$). Note that $n=0$ for $\mathcal{L}_0/L \leq 0.1$, and it becomes nearly 6 for $\mathcal{L}_0/L \simeq 100$.

from 0 m to 250 m for both models (Kolmogorov and von Karman with $\mathcal{L}_0 = 20$ m).

This result shows that Kolmogorov phase screens always need the same number of subharmonics for a given requested accuracy, whatever the screen length L. A good number seems to be 9, after which there is a saturation effect of the attainable accuracy. For von Karman screens, it is obviously dependent on $\mathcal{L}_0$, but it is also obvious that von Karman screens always need fewer subharmonics to be added for the same ratio α_D.

3. Zernike-Polynomials-Based Method Using Jacobi Polynomials

When the Zernike polynomials method is used, the phase screens $\varphi(r,\theta)$ are generated as the sum of the first $N = (j_{\max} - 1)$ Zernike polynomials $Z_j(\frac{r}{R},\theta)$:

$$\varphi(r,\theta) = \sum_{j=2}^{j_{\max}} c_j Z_j\left(\frac{r}{R},\theta\right), \tag{12}$$

where r and θ are the polar coordinates on the pupil of radius R. The Zernike polynomial is expressed as a multiplication between a radial component $R_n^m(\rho)$ and an azimuthal component $[\sin(m\theta)$ or $\cos(m\theta)]$, as described by Noll [6], where n and m represent the radial and azimuthal orders, respectively. The piston term $(j = 1)$ is not considered because the point-spread function is not depending on it.

For each phase screen realization, an independent set of the coefficients c_j has to be obtained with the correct statistics stated by the $N \times N$ covariance matrix [7]: $C_{j,j'} = \langle c_j c_{j'} \rangle$. Only a small portion of the matrix has no zero elements, so sparse matrix algorithms can be used when implementing the method in order to allocate less memory and increase the code speed [8].

To obtain good behavior of the phase screens at high spatial frequencies, $j_{\max}$ must be large enough to take into account polynomials with a number of oscillations over the pupil diameter comparable to the chosen sample (usually at least two or three pixels per r_0). In these conditions, $j_{\max}$ can be as large as, for example, a few thousands in the visible for 8 m class telescopes.

The usual definition formula for the Zernike radial component $R_n^m(\rho)$ [6] fails when computing polynomials with high radial order, because the formula involves differences between large terms that should produce a result around the unit value. For instance, when double precision arithmetic is used, numerical instability starts to be evident for $j \gtrsim 1030$, when the largest coefficients of the radial component of the corresponding Zernike polynomial have values around 1×10^{15}, close to the inverse of the floating point precision.

To overcome this problem, we developed an algorithm based on an alternative definition for $Z_j(\rho,\theta)$, involving the relationship between the Zernike radial component $R_n^m(\rho)$ and the Jacobi polynomials $P_k^{\alpha,\beta}(x)$ [9] and using both the Jacobi polynomials definition and their recurrence relationship from Magnus *et al.* [10], as follows:

$$R_n^m(\rho) = \rho^m P_{(n-m)/2}^{(0,m)}(2\rho^2 - 1), \tag{13}$$

where

$$P_k^{\alpha,\beta}(x) = \frac{[C_1 P_{k-1}^{\alpha,\beta}(x) - C_2 P_{k-2}^{\alpha,\beta}(x)]}{C_0}, \tag{14}$$

$$P_0^{\alpha,\beta}(x) = 1, \tag{15}$$

$$P_1^{\alpha,\beta}(x) = (\alpha + \beta + 2)/2x + (\alpha - \beta)/2, \tag{16}$$

$$C_0 = 2k(\alpha + \beta + k)(\alpha + \beta + 2k - 2), \tag{17}$$

$$C_1 = (2k + \alpha + \beta - 2)(2k + \alpha + \beta - 1)(2k + \alpha + \beta)x + (\alpha^2 - \beta^2)(2k + \alpha + \beta - 1), \tag{18}$$

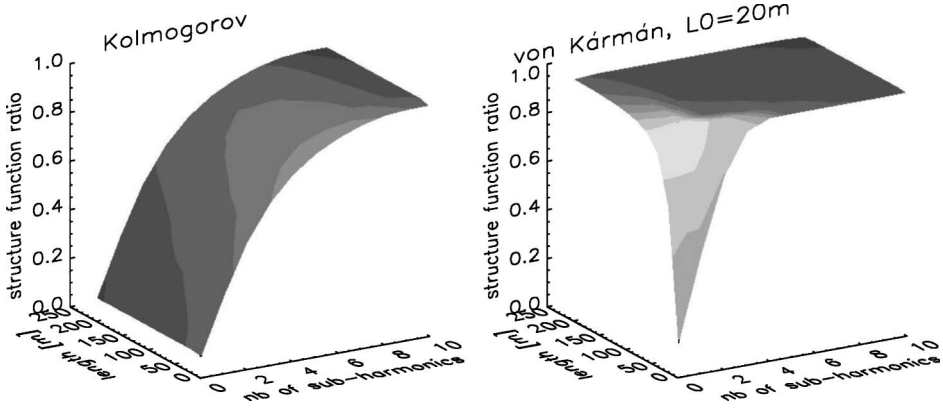

Fig. 2. Structure function ratio α_D versus the number of subharmonics to be added n and the screen physical length L, for the Kolmogorov model (left) and the von Karman model (right, with $\mathcal{L}_0 = 20$ m).

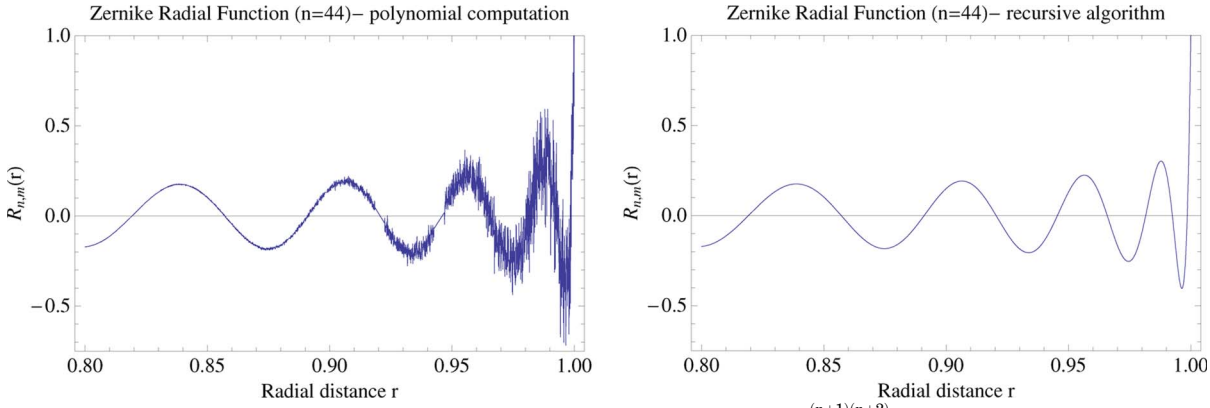

Fig. 3. (Color online) Zernike radial function computed for $n = 44$ (corresponding to $j = \frac{(n+1)(n+2)}{2} = 1035$). Left: polynomial computation (classical method) result. Right: recursive algorithm (method presented here) result.

$$C_2 = 2(k+\alpha-1)(2k+\alpha+\beta)(k+\beta-1). \qquad (19)$$

Figure 3 illustrates the problem of precision exposed above, showing the result of both methods (the classical method and the method proposed here) for $j = 1030$, as a function of the (normalized) radial distance, while Fig. 4 shows an example of phase screen generated using our method. The structure function (computed from 400 phase screens of 256 × 256 pixels, i.e., 8 m × 8 m, with $j_{\max} = 4186$ and $r_0 = 1$ m) is here, again, in very good agreement with the theoretical method (Kolmogorov here).

4. Summary and Concluding Remarks

We have described new solutions permitting to overcome the well-known problems encountered when employing the two main classical methods for numerical modeling of atmospherically perturbed phase screens, namely the FFT-based and the Zernike-based methods.

Concerning the FFT-based numerical method, we have defined two criteria permitting us to precisely chose the number of subharmonics needed in order to overcome the lack of low frequencies inherent to this method and show their practical efficiency.

Concerning the Zernike-based method, we have developed an algorithm based on an alternative definition of the Zernike polynomials, involving the recurrence definition of the Jacobi polynomials as well as the relationship between the Zernike polynomials and the Jacobi polynomials.

The whole code implemented for and used in this paper (the FFT-based method with subharmonics adding and aid-to-decision tools, making use of the two criteria defined in this paper and permitting us to choose the number of subharmonics to be added, and the Zernike-polynomials-based method with the use of Jacobi polynomials) has been implemented within the freely distributed software package CAOS [11] and, more precisely, within the ATM (which stands for "ATMosphere building") module of this AO-dedicated package. CAOS is itself developed within the homonymic CAOS problem-solving environment (PSE) [12]. Both CAOS and the CAOS PSE are downloadable from the dedicated website http://fizeau.unice.fr/caos.

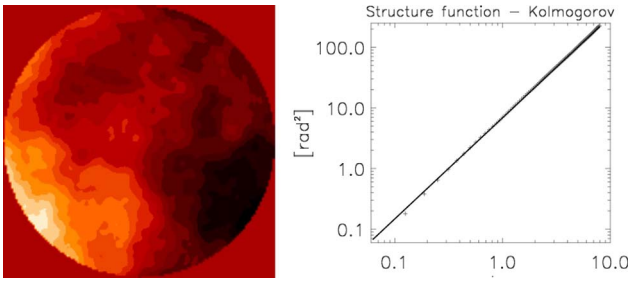

Fig. 4. (Color online) Left: example of a Kolmogorov/Zernike phase screen. Right: the associated theoretical (straight line) and simulated (crosses) structure functions, in logarithmic scale.

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