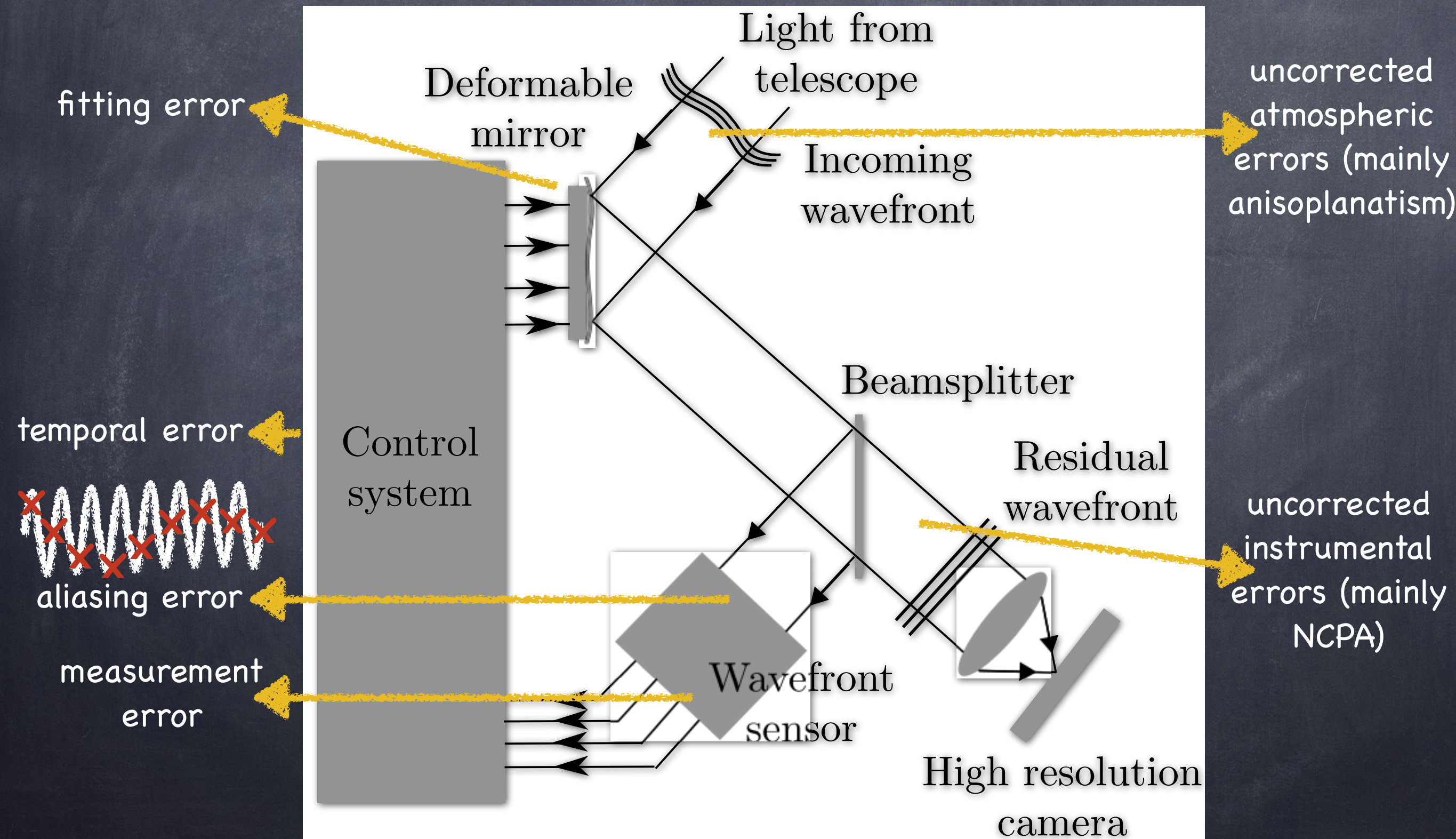


Post-AO error budget - 1



Post-AO error budget - 2

Global error (variance on the corrected phase/wavefront)

$$\sigma_{\text{post-AO}}^2 = \sigma_{\text{atm.}}^2 + \sigma_{\text{AO syst.}}^2 + \sigma_{\text{others}}^2$$

- Error term due to turbulent atmosphere alone
- Residual error of the AO system itself
- Other error terms...

Post-AO error budget - 3

Other errors

$$\sigma_{\text{others}}^2 = \sigma_{\text{NCPA}}^2 + \sigma_{\text{calib.}}^2 + \dots$$

- Error term due to Non-Common Path Aberrations
- Error term due to the calibration of the AO system
- etc.

Post-AO error budget - 4

Error due to atmospheric turbulence alone

$$\sigma_{\text{atm.}}^2 = \sigma_{\text{aniso}}^2 + \sigma_{\text{scint.}}^2 + \sigma_{\text{diff.}}^2 + \sigma_{\text{chrom.}}^2$$

- Error term due to anisoplanatism
- Error term due to scintillation
- Error term due to diffractive effects
- Error term due to differential refraction

Post-AO error budget - 5

Residual error of the AO system itself

$$\sigma_{\text{AO syst.}}^2 = \sigma_{\text{fitt.}}^2 + \sigma_{\text{meas.}}^2 + \sigma_{\text{alias.}}^2 + \sigma_{\text{temp.}}^2 + \dots$$

$$\dots + \sigma_{\text{LGS}}^2 + \sigma_{\text{MCAO}}^2$$

- Fitting error (due to spatial under-sampling of the DM)
- Measurement error (due to photon noise, RON, etc. - WFS)
- Aliasing error (due to spatial under-sampling of the WFS)
- Temporal error (due to finite temporal bandwidth of the whole system)
- Specific errors of the LGS
- Specific errors of the MCAO (et similia)

Post-AO error budget - 6

Fitting error

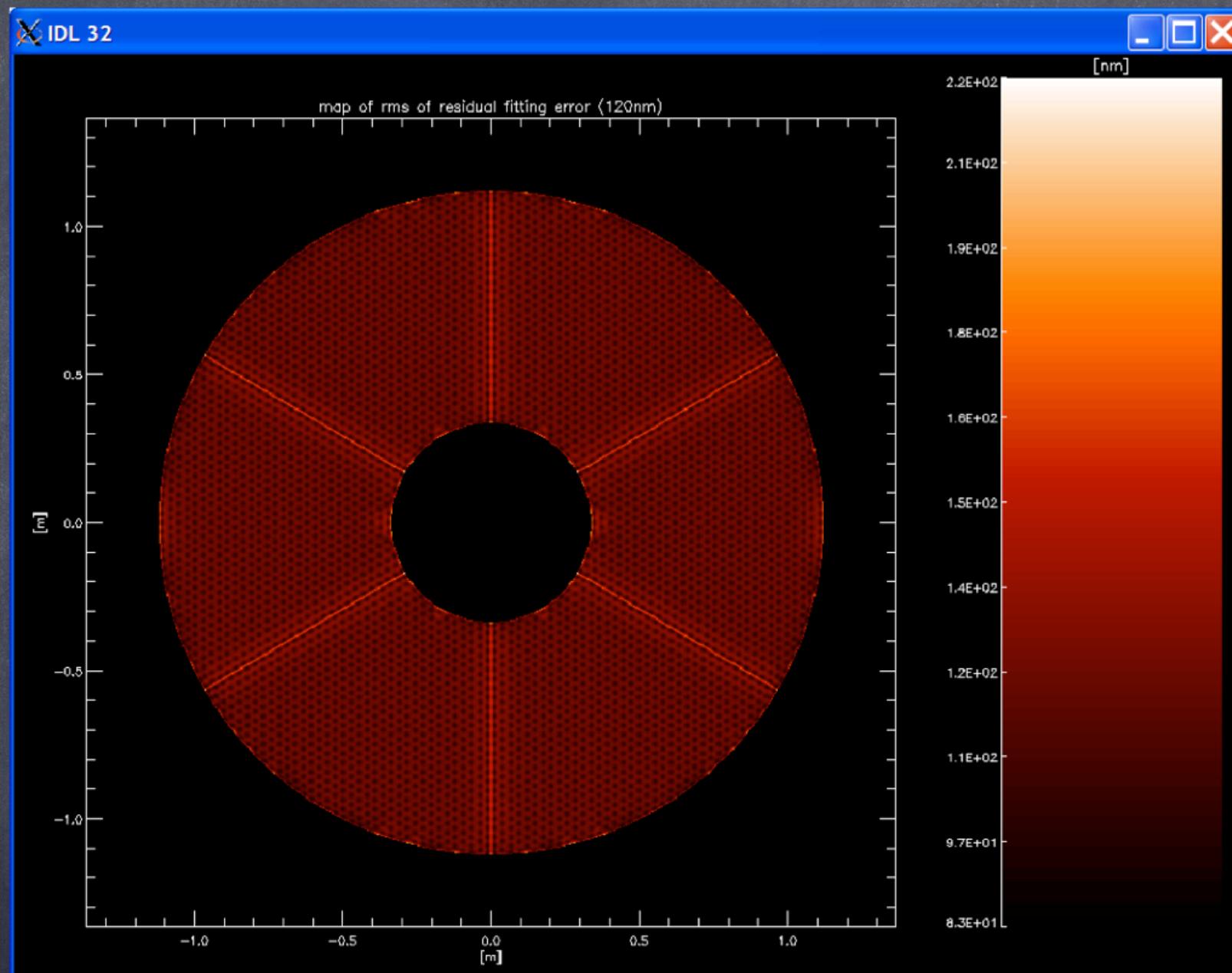
$$\sigma_{\text{AO syst.}}^2 = \sigma_{\text{fitt.}}^2 + \sigma_{\text{meas.}}^2 + \sigma_{\text{alias.}}^2 + \sigma_{\text{temp.}}^2 + \dots$$

$$\sigma_{\text{fitt.}}^2 \propto \left(\frac{d}{r_0} \right)^{\frac{5}{3}}$$

$$\sigma_{\text{fitt.}}^2 \approx 0.23 \left(\frac{d}{r_0} \right)^{\frac{5}{3}}$$

Reduce the fitting error $\Leftrightarrow$ increase the number of actuators of the DM...

Post-AO error budget - 7



(Noll: residual error - 1)

If, instead of having actuators, one could have a mirror capable of forming perfect Zernike polynomials, one would have (admitting that atmosphere exactly follows a Kolmogorov model):

$$\Delta_J \simeq 0.2944 J^{-\sqrt{3}/2} \left(\frac{D}{r_0} \right)^{5/3}, J \geq 20$$

Hence, in meters (assuming to point to zenith):

$$\sigma_J[m] \simeq \frac{\lambda}{2\pi} \sqrt{\Delta_J} \simeq 0.352 J^{-\sqrt{3}/4} D^{5/6} \left(\int_0^\infty C_n^2(z) dz \right)^{1/2}$$

With, thanks to Maréchal's approximation:

$$S \simeq \exp \{ -\Delta \}$$

(Noll: residual error - 2)

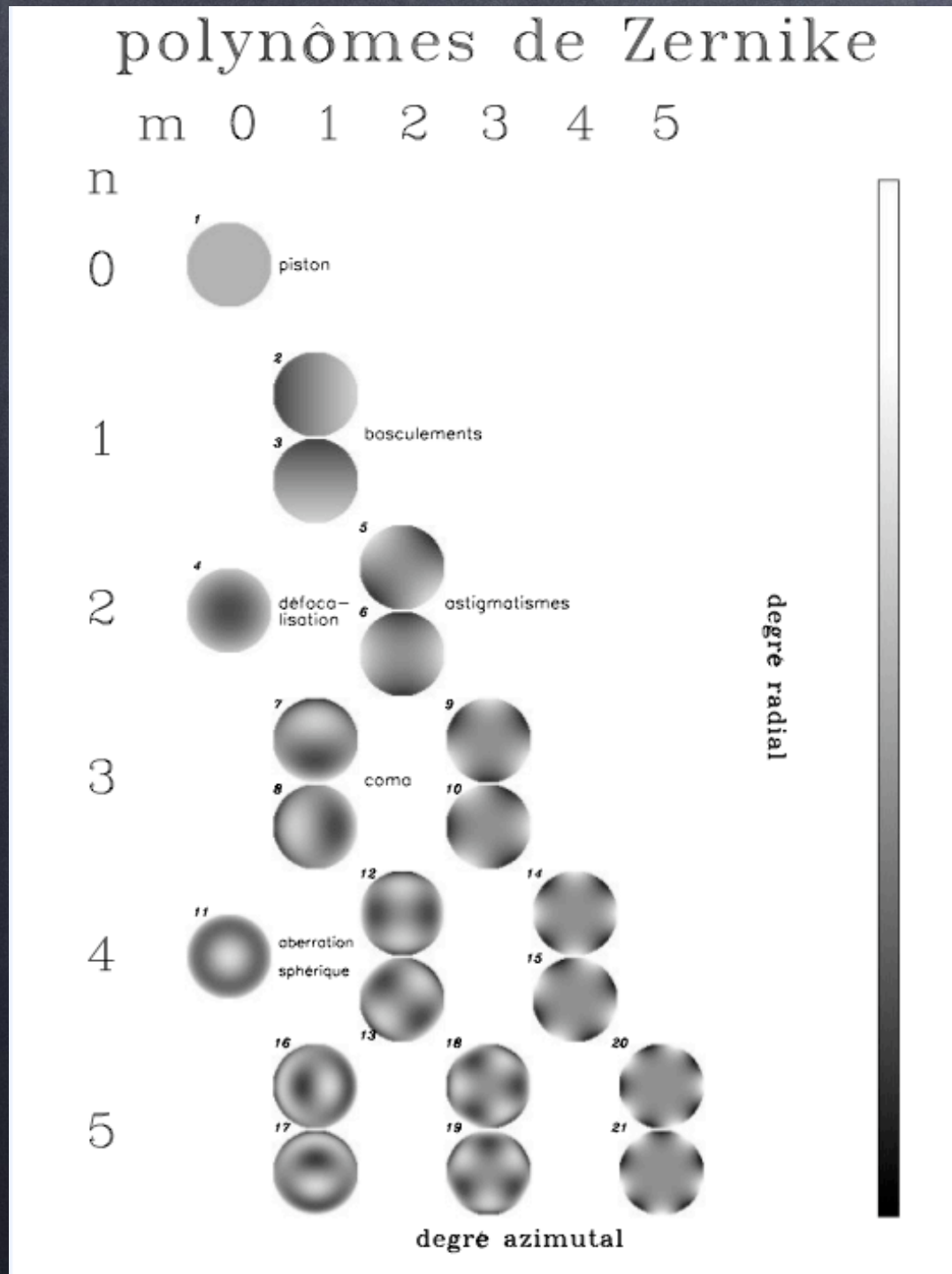
TABLE IV. Zernike-Kolmogoroff residual errors (Δ_J). (D is the aperture diameter.)

$\Delta_1 = 1.0299 (D/r_0)^{5/3}$	$\Delta_{12} = 0.0352 (D/r_0)^{5/3}$
$\Delta_2 = 0.582 (D/r_0)^{5/3}$	$\Delta_{13} = 0.0328 (D/r_0)^{5/3}$
$\Delta_3 = 0.134 (D/r_0)^{5/3}$	$\Delta_{14} = 0.0304 (D/r_0)^{5/3}$
$\Delta_4 = 0.111 (D/r_0)^{5/3}$	$\Delta_{15} = 0.0279 (D/r_0)^{5/3}$
$\Delta_5 = 0.0880 (D/r_0)^{5/3}$	$\Delta_{16} = 0.0267 (D/r_0)^{5/3}$
$\Delta_6 = 0.0648 (D/r_0)^{5/3}$	$\Delta_{17} = 0.0255 (D/r_0)^{5/3}$
$\Delta_7 = 0.0587 (D/r_0)^{5/3}$	$\Delta_{18} = 0.0243 (D/r_0)^{5/3}$
$\Delta_8 = 0.0525 (D/r_0)^{5/3}$	$\Delta_{19} = 0.0232 (D/r_0)^{5/3}$
$\Delta_9 = 0.0463 (D/r_0)^{5/3}$	$\Delta_{20} = 0.0220 (D/r_0)^{5/3}$
$\Delta_{10} = 0.0401 (D/r_0)^{5/3}$	$\Delta_{21} = 0.0208 (D/r_0)^{5/3}$
$\Delta_{11} = 0.0377 (D/r_0)^{5/3}$	

$$\Delta_J \sim 0.2944 J^{-3/2} (D/r_0)^{5/3} \quad (\text{For large } J)$$

(From Noll, JOSA 66, 1976)

(Noll: residual error - 3)

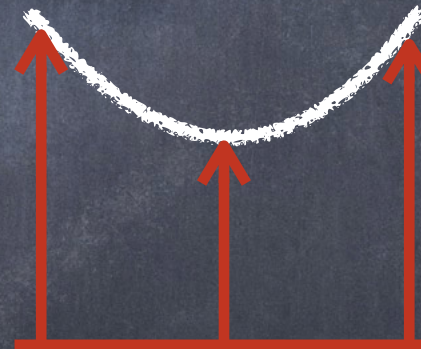


Example of the tip-tilt



=> needs 2 actuators (linearly)
for the 1st radial order

Example of the defocus



=> needs 3 actuators
(2nd radial order)

radial order $N \Leftrightarrow N+1$ actuators (DM)
 $\Leftrightarrow N$ elements of analysis (sub-apertures)
 $\Leftrightarrow$ « $N \times N$ system »

(Noll: residual error - 4)

Exercise 1: Compute the Noll error and then the corresponding maximum Strehl ratio in J ($1.25 \mu\text{m}$) for a 10×10 AO system.

[knowing that: $r_0@500\text{nm} = 10\text{cm}$ and $D = 1\text{m}$, and that nb of Z modes corrected = $(n+1)(n+2)/2$ piston included, n being the radial order]

(Noll: residual error - 5)

$$J = (N+1) (N+2) / 2$$

Here: $N=10 \Rightarrow J = 11 \times 12 / 2 = 66$ Zernike modes (piston included)

And: $r_0^{1.25\mu\text{m}} = 0.1 (1.25/0.5)^{1.2} \approx 0.3 \text{ m}$

$$\Delta_J \simeq 0.2944 J^{-\sqrt{3}/2} \left(\frac{D}{r_0} \right)^{5/3}, J \geq 20$$

Here: $\Delta_J = 0.2944 J^{-\sqrt{3}/2} (1/0.3)^{5/3} \approx 0.05816 \text{ rad}^2$ (for $\lambda=1.25\mu\text{m}$)

$$S = \exp(-\Delta_J)$$

Here: $S \approx \exp(-0.05816) \approx 0.94$, i.e. $\approx 94\%$ Strehl ratio (at $\lambda=1.25\mu\text{m}$).

(Noll: residual error - 6)

```
; call with: IDL> @Exo2
Diam  =1.0
r0    =0.3
N     = 10

J = (N+1)*(N+2)/2-X
Noll = .2944*J^(-sqrt(3)/2)*(Diam/r0)^(5./3)
S = exp(-Noll)
; see result with: IDL> print, S
```

batch: all the variables defined are accessible

```
; call with: IDL> .rn Exo2_main
Diam  =1.0
r0    =0.3
N     = 10

J = (N+1)*(N+2)/2-X
Noll = .2944*J^(-sqrt(3)/2)*(Diam/r0)^(5./3)
S = exp(-Noll)

end
; see result with: IDL> print, S
```

main: idem (« .rn » : run new)

```
; call with: IDL> .rn Exo2_proc
; IDL> Exo2_proc, Diam, r0, N, S
; with, e.g: Diam=1.0, r0=0.3, N=10, S undefin
pro Exo2_proc, Diam, r0, N, S

J = (N+1)*(N+2)/2-X
Noll = .2944*J^(-sqrt(3)/2)*(Diam/r0)^(5./3)
S = exp(-Noll)

end
; see result with: IDL> print, S
```

procedure: the input/output parameters are accessible, not the variables defined inside the procedure

```
; call with: IDL> .rn Exo2_func
; IDL> print, Exo2_func(Diam, r0, N)
; with, e.g: Diam=1.0, r0=0.3, N=10
function Exo2_func, Diam, r0, N

J = (N+1)*(N+2)/2-X
Noll = .2944*J^(-sqrt(3)/2)*(Diam/r0)^(5./3)
S = exp(-Noll)

return, S
end
```

function: no output parameters, variables defined not accessible, results of the function returned.

Voir aussi la fonction noll.pro dans CAOS !!

(Noll: residual error - 6+)

```
function noll, mode, D, r0
;+
; computes the Noll residual variance [rad^2], noll.
; => wf rms [m] = sqrt(noll)*lambda/(2*PI),
; where lambda = wavelength [m] for which r0 is defined.
; => Strehl (for wavelength for which r0 is defined, lambda) = exp{-noll}.
; => Strehl (for another wavelength, lambda_i) = exp{-noll*(lambda/lambda_i)^2}.
;
; use: noll = noll(mode, D, r0)
; where:
; mode = Zernike (starting from piston=mode #0) untill which the
; ideal correction is made,
; D = diameter of the telescope [m],
; r0 = Fried parameter [m],
; noll = residual variance [rad^2].
;
; Marcel Carillet (Lagrange, UNS/CNRS/OCA) [marcel.carillet@unice.fr]
; for CAOS Library 5.4, 22 March 2012.
;-

if mode eq 0 then noll = 1.0299 ; piston (Z1)
if mode eq 1 then noll = .582   ; Zernike tip (Z2)
if mode eq 2 then noll = .134   ; Zernike tilt (Z3)
if mode eq 3 then noll = .111   ; Defocus (Z4)
if mode eq 4 then noll = .0880  ; Astigmatism 1 (Z5)
if mode eq 5 then noll = .0648  ; Astigmatism 2 (Z6)
if mode eq 6 then noll = .0587  ; Coma 1 (Z7)
if mode eq 7 then noll = .0525  ; Coma 2 (Z8)
if mode eq 8 then noll = .0463  ; Z9
if mode eq 9 then noll = .0401  ; Z10
if mode eq 10 then noll = .0377 ; 3rd order spherical (Z11)
if mode eq 11 then noll = .0352 ; Z12
if mode eq 12 then noll = .0328 ; Z13
if mode eq 13 then noll = .0304 ; Z14
if mode eq 14 then noll = .0279 ; Z15
if mode eq 15 then noll = .0267 ; Z16
if mode eq 16 then noll = .0255 ; Z17
if mode eq 17 then noll = .0243 ; Z18
if mode eq 18 then noll = .0232 ; Z19
if mode eq 19 then noll = .022  ; Z20
if mode eq 20 then noll = .0208 ; Z21
if mode gt 20 then noll = .2944*(mode+1.)^(-sqrt(3)/2.)

return, noll*(D/r0)^(5./3)
end
```

(Noll: residual error - 7)

Remark:

One will often rather start from the evaluation of the *rms* on the residual wavefront ($\sigma[\text{m}]$)...

$$\sigma[\text{m}] = \lambda/2\pi \sqrt{\Delta_J}$$

Here: $\sigma[\text{m}] = \textit{rms}$ of the residual wf
 $= 1.25 \cdot 10^{-6} / 2\pi \sqrt{\Delta_J} \approx 48 \text{ nm}$

And then:

$$S_\lambda = \exp\{-(2\pi/\lambda \sigma[\text{m}])^2\}$$

Here: $S_{1.25\mu\text{m}} = \exp(-(2\pi/1.25\mu\text{m} \sigma[\text{m}])^2) \approx 0.94$.

(Noll: residual error - 8)

Exercise 2: Which mirror configuration for a (minimum, other errors excluded) goal Strehl ratio of 30% in band J ($1.25\mu\text{m}$) ?

[knowing that: $r_0@500\text{nm} = 12\text{cm}$ and $D = 8\text{m}$, and that nb of Z modes corrected = $(n+1)(n+2)/2$ piston included, n being the radial order]

-> For next session: solve this exercice !!
(IDL batch, and also function)

Post-AO error budget - 8

Aliasing error

$$\sigma_{\text{AO syst.}}^2 = \sigma_{\text{fitt.}}^2 + \sigma_{\text{meas.}}^2 + \sigma_{\text{alias.}}^2 + \sigma_{\text{temp.}}^2 + \dots$$

$$\sigma_{\text{alias.}}^2 \propto \left(\frac{d}{r_0} \right)^{\frac{5}{3}}$$

$$\sigma_{\text{alias.}}^2 \approx 0.08 \left(\frac{d}{r_0} \right)^{-\frac{5}{3}}$$

Reduce the aliasing error $\Leftrightarrow$ increase the number of sensing elements (Shack-Hartmann sub-apertures, pixels of the pyramid) within the WFS

Post-AO error budget - 9

Temporal error

$$\sigma_{\text{AO syst.}}^2 = \sigma_{\text{fitt.}}^2 + \sigma_{\text{meas.}}^2 + \sigma_{\text{alias.}}^2 + \sigma_{\text{temp.}}^2 + \dots$$

$$\sigma_{\text{temp.}}^2 \propto \left(\frac{\Delta t_{\text{AO}}}{\tau_0} \right)^{\frac{5}{3}}$$

$$\sigma_{\text{temp.}}^2 \approx \left(\frac{\Delta t_{\text{AO}}}{\tau_0} \right)^{\frac{5}{3}}$$

Reduce the temporal error $\Leftrightarrow$ make a faster system
(exposure time of the WFS, computing time for the
wavefront reconstruction, actuating time for the DM)

Post-AO error budget - 10

Measurement error

$$\sigma_{\text{AO syst.}}^2 = \sigma_{\text{fitt.}}^2 + \sigma_{\text{meas.}}^2 + \sigma_{\text{alias.}}^2 + \sigma_{\text{temp.}}^2 + \dots$$

$$\sigma_{\text{mes.}}^2 = \sigma_{\text{phot.}}^2 + \sigma_{\text{RON}}^2 + \dots$$

The measurement error has many origins:

- photon noise
- read-out noise (RON)
- dark-current noise
- sky background and possibly instrumental background
- in case of EMCCD: (almost) no RON but additional noises (exotic dark, « excess noise factor » => Gamma-distributed noise)

Post-AO error budget - 11

Photon noise error term

$$\sigma_{\text{AO syst.}}^2 = \sigma_{\text{fitt.}}^2 + \sigma_{\text{meas.}}^2 + \sigma_{\text{alias.}}^2 + \sigma_{\text{temp.}}^2 + \dots$$

$$\sigma_{\text{mes.}}^2 = \sigma_{\text{phot.}}^2 + \sigma_{\text{RON}}^2 + \dots$$

$$\sigma_{\text{phot.}}^2 \propto \frac{1}{N_{\text{phot.}}}$$

$$\sigma_{\text{phot.}}^2 \propto \frac{1}{N_{\text{phot.}} r_0^2 \tau_0}, \tau_0 \propto r_0$$

(with N_{phot} =number of photons/exposure time/subaperture)

Reduce the photon noise error term $\Leftrightarrow$

1- reduce the number of WFS elements $\Rightarrow$ increase the aliasing error !!

2- increase the exposure time $\Rightarrow$ increase the temporal error !!

Post-AO error budget - 12

Read-out noise error term

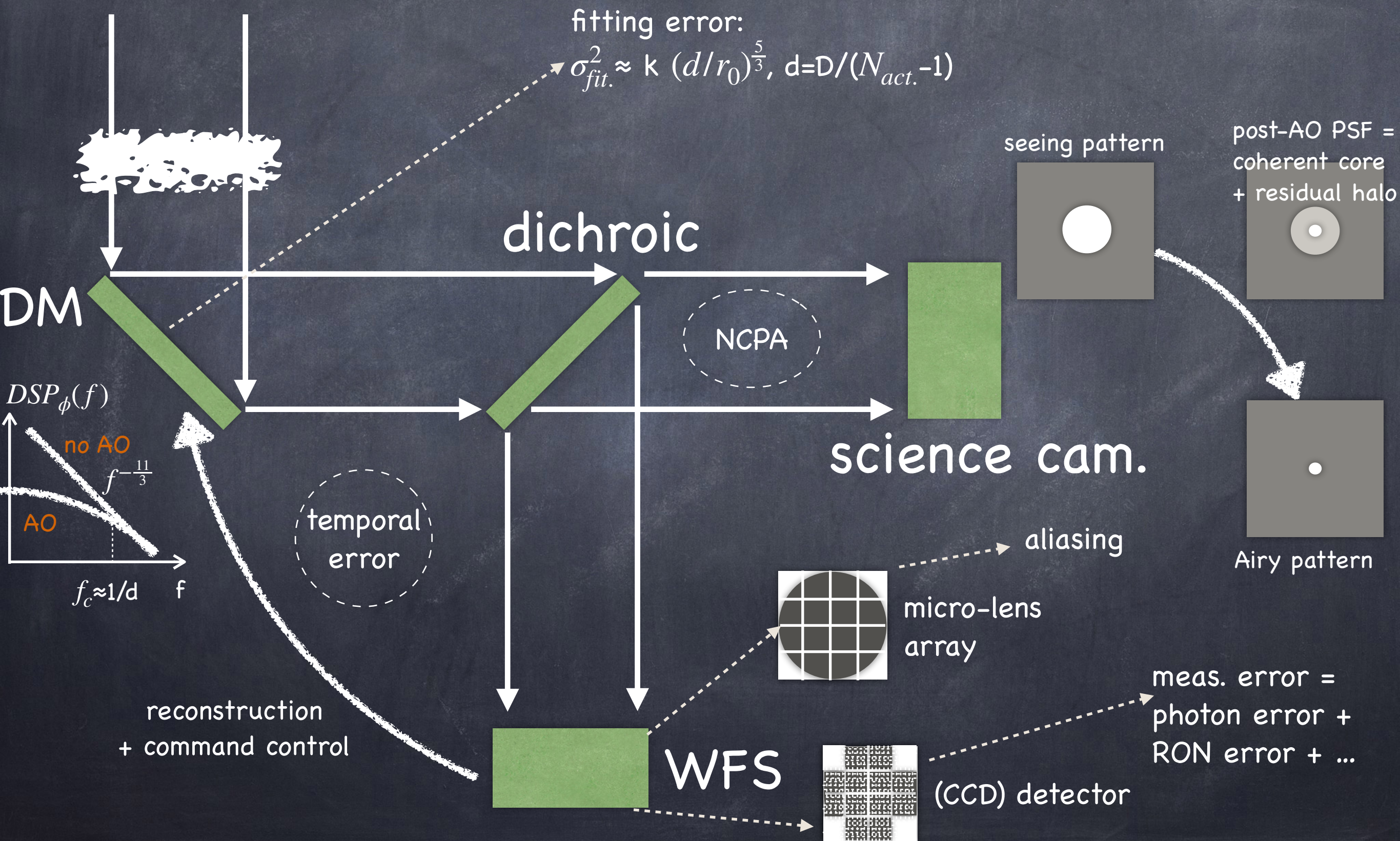
$$\sigma_{\text{AO syst.}}^2 = \sigma_{\text{fitt.}}^2 + \sigma_{\text{meas.}}^2 + \sigma_{\text{alias.}}^2 + \sigma_{\text{temp.}}^2 + \dots$$

$$\sigma_{\text{mes.}}^2 = \sigma_{\text{phot.}}^2 + \sigma_{\text{RON}}^2 + \dots$$

$$\sigma_{\text{lect.}}^2 \propto \frac{\sigma_e}{N_{\text{phot.}}^2}$$

Reduce the RON error term $\Leftrightarrow$

- 1- reduce the number of WFS elements $\Rightarrow$ increase the aliasing error !!
- 2- increase the exposure time $\Rightarrow$ increase the temporal error !!
- 3- but also: reduce the impact of RON $\Rightarrow$ use of EMCCDs...



Post-AO error budget - 14

Generic case: observe as much sources as possible

Problem: most sources are (obviously!) too faint

1- find and use brighter NGS nearby...

=> anisoplanatic error !

=> use more than one brighter NGS nearby...

=> multi-reference AO system (GLAO, MCAO, MOAO)

=> yes, but: specific errors !

=> limited quality of correction

2- create a brighter source (LGS)

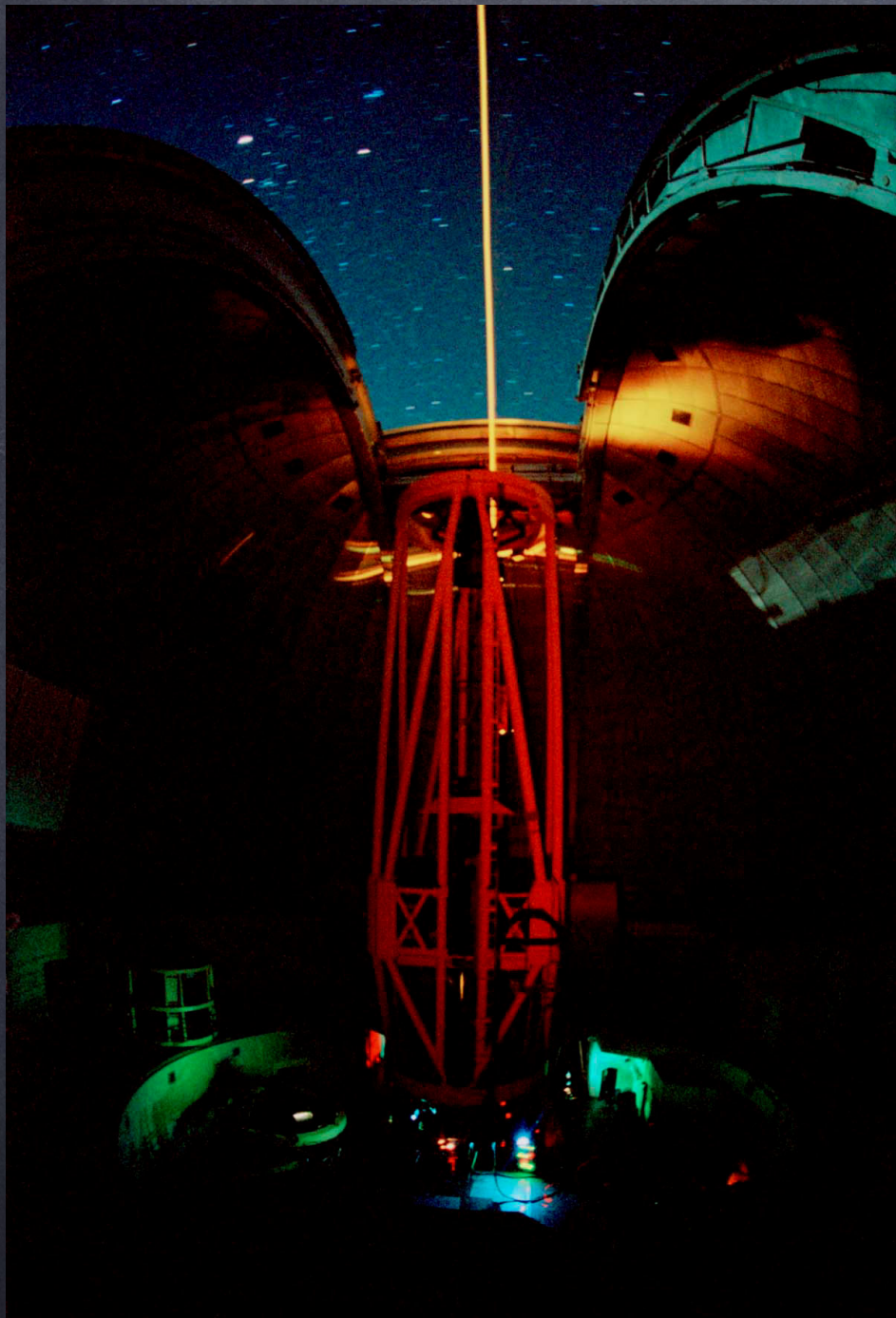
=> 100% sky coverage

=> yes, but here again: specific errors !

=> limited quality of correction

Laser Guide Star AO - 1

[Harper 1982, Foy & Labeyrie 1985]



(Keck Observatory)

+ https://www.youtube.com/watch?v=3BpT_tXYy_I

Laser Guide Star AO - 2

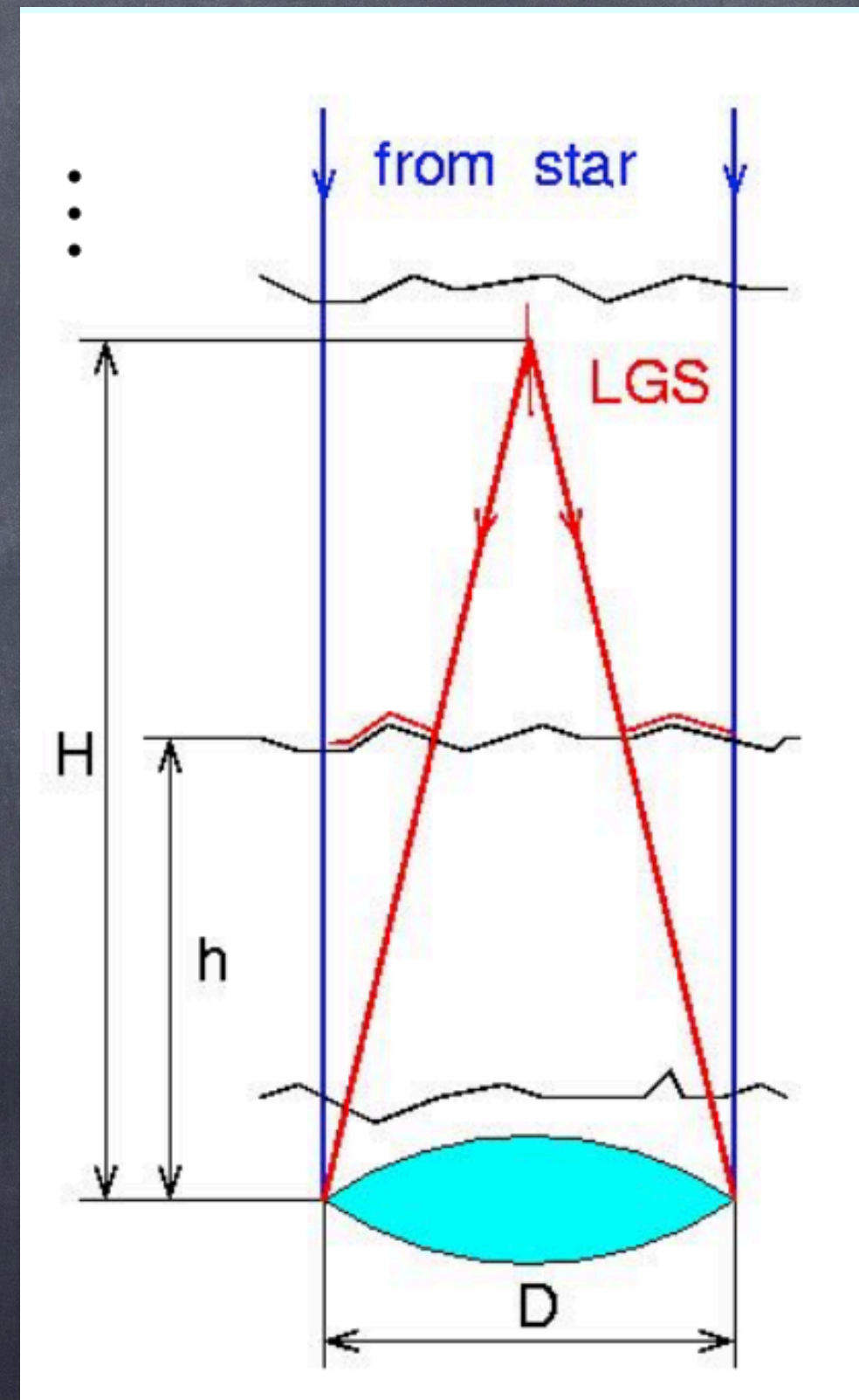
Specific LGS errors:

- 1- Cone effect (focus aniso.)
- 2- Tip-tilt indetermination
- 3- Perspective elongation

In addition (Na LGS case):

- Rayleigh backscattering until $\sim 35\text{km}$,
- variation of the Na layer column density (seasonally: a factor $\sim 5!$ but also short time) and of the centroid height.

Science w/LGS: solar system bodies, YSO (circumstellar material), brown dwarves, Galactic novae, starburst galaxies, AGN, radio galaxies, gravitational lenses, clusters of galaxies, etc.



Laser Guide Star AO - 3

1- Cone effect

- LGS formed at finite altitude (H_{LGS}), with $H_{\text{LGS}}=90\text{-}100\text{km}$ for a Sodium (Na) LGS, and $H_{\text{LGS}}=10\text{-}20\text{km}$ for a Rayleigh LGS.
- If $h_{\text{layer}} > H_{\text{LGS}}$, the turbulent layer is not sensed.
- Even when $h_{\text{layer}} < H_{\text{LGS}}$, the outer portions of the turbulent layer are not sensed (with if $h_{\text{layer}} \neq 0$).
- The turbulent layer at altitude h_{layer} is also sampled differently by the LGS and the observed object at infinity (if $h_{\text{layer}} \neq 0$)
=> the laser beam diameter is reduced by a factor $(1-h/H)$
=> there is a differential « stretching » between the laser wavefront and the object wavefront, and:

$$\sigma_{\text{cone}}^2 = \left(\frac{D}{d_0} \right)^{\frac{2}{3}}, \text{ with } d_0 \sim 2.91 \theta_0 H_{\text{LGS}}$$

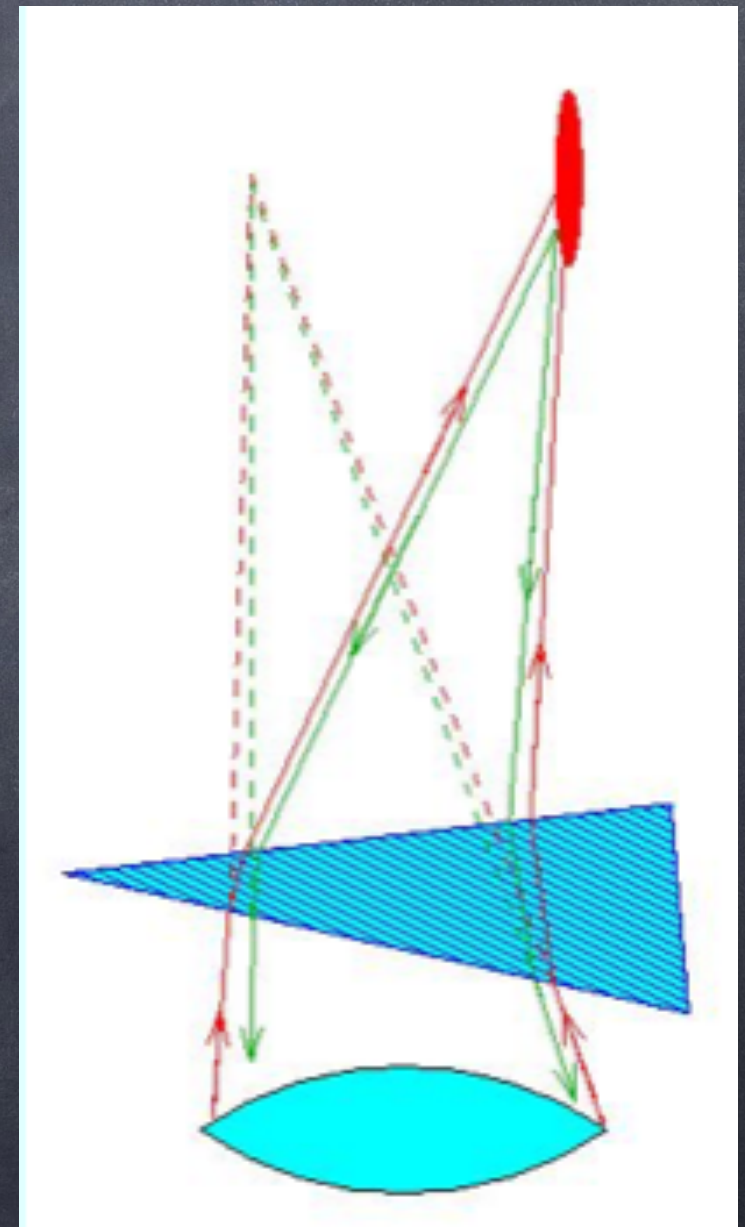
For example: $\theta_0 \sim 2.5''$, $H=90\text{km}$ => $d_0 \sim 3.2\text{m}$.

- Multiple beacons => focal anisoplanatism decreases !
[see Nature cover 01/2000, Ragazzoni]

Laser Guide Star AO - 4

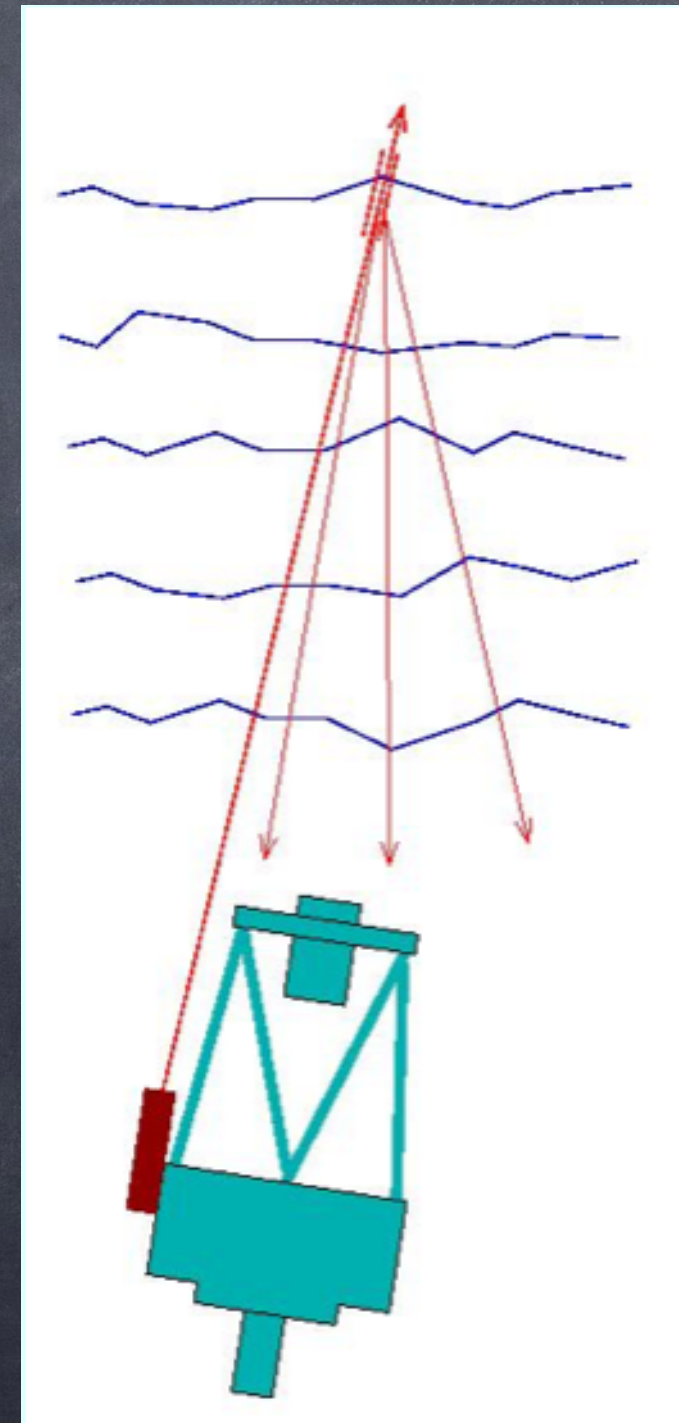
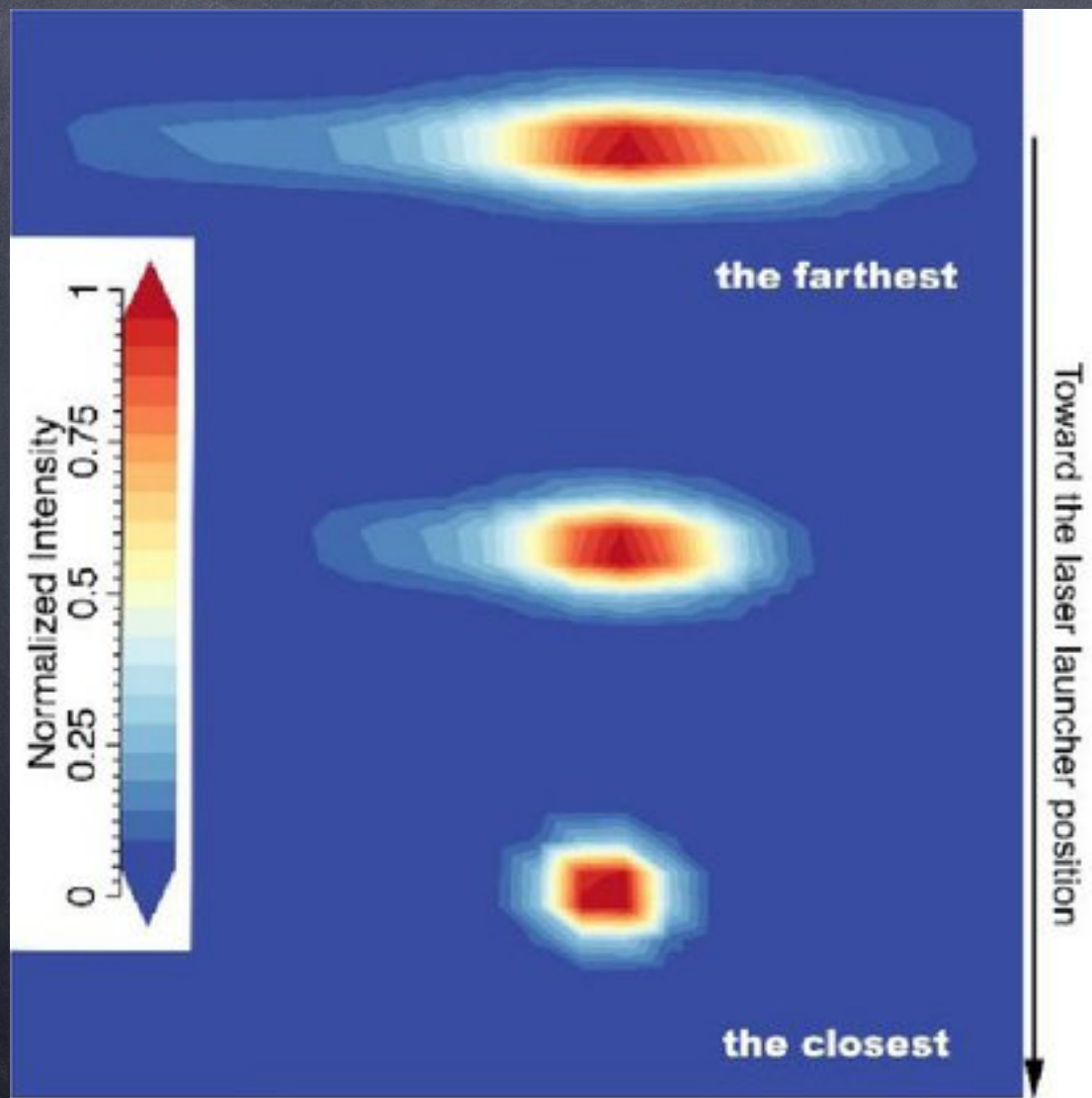
2- Tip-tilt indetermination => tip-tilt anisoplanatism error
(= anisokinetism error)

$$\sigma_{\text{TT aniso.}}^2 \sim 0.1 \left(\frac{\theta}{\theta_0} \right)^2 \left(\frac{D}{r_0} \right)^{-\frac{1}{3}}$$



Laser Guide Star AO - 5

3- Perspective elongation => SH spots elongation
(Na: 10–15km@90–100km, 589nm,
Rayleigh: ~2km@10–20km, 355nm)



Quality of correction ? - 1

$$S = \frac{I_{\text{post AO}}[0, 0]}{I_{\text{perfect}}[0, 0]}$$

where $I[0,0]$ is the intensity of the PSF at the optical center of the field (K. Strehl, Zeit. Instrumentkde 22, 213 (1902)).

$$S \simeq \exp\{-\sigma_{\text{post AO}}^2\}$$

in the framework of the Maréchal's approximation, where the variance (in radians²) is supposed to be small enough...

→ see also page 4 of Carbillet et al., MNRAS (2017)

Quality of correction ? - 2

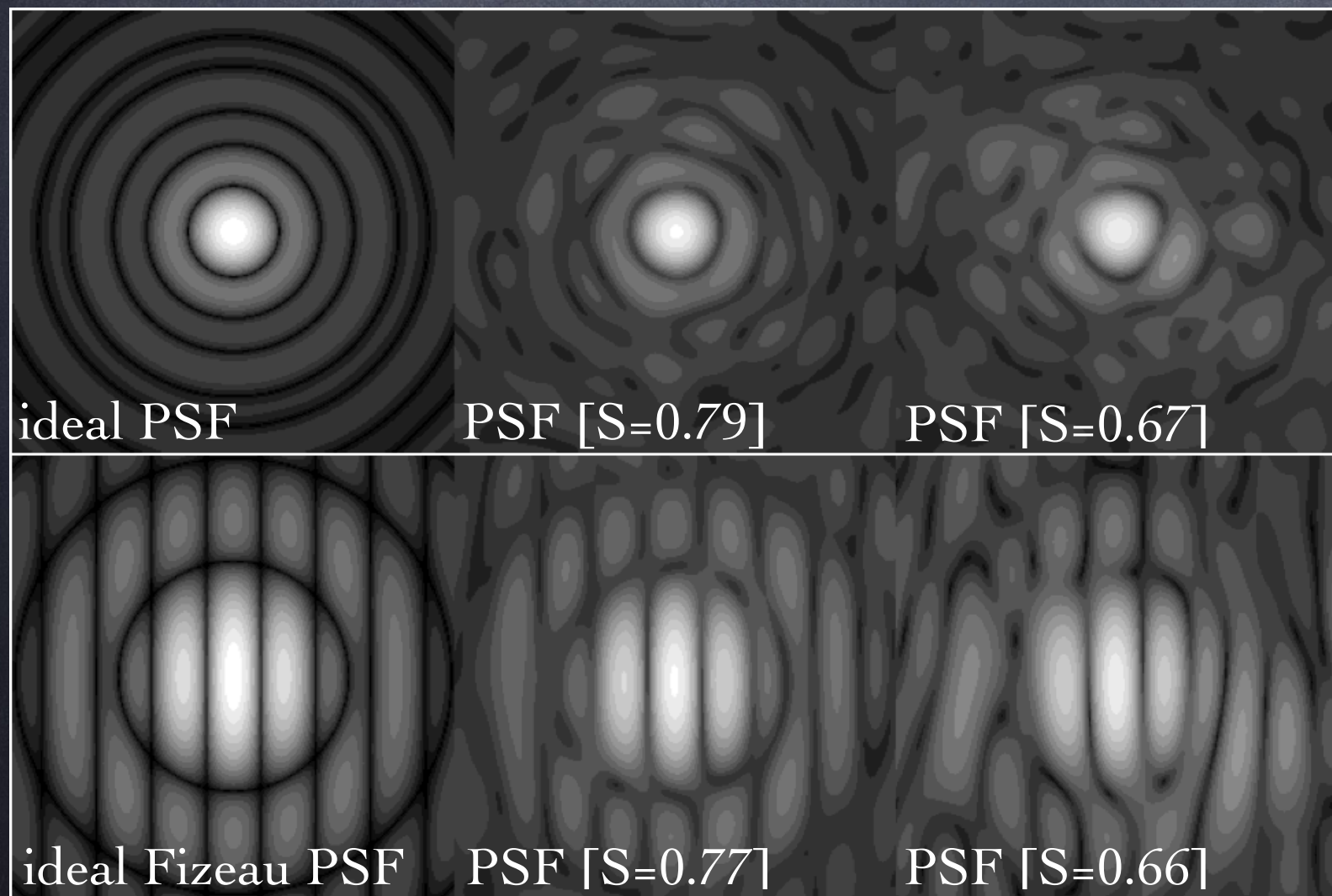
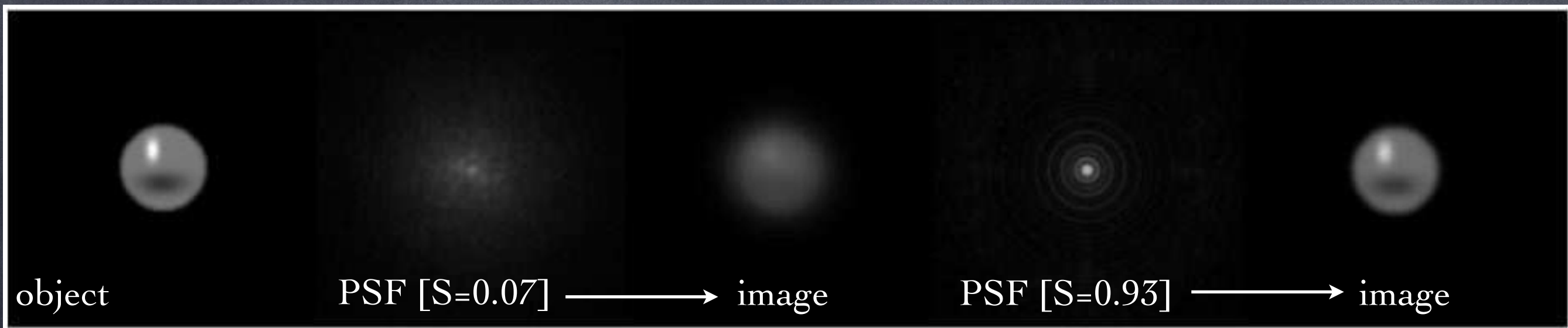
Approximation which neglects tip-tilt: ratio of the maxima

Ratio of the values at the centre of the image $\approx$ ratio of the OTF (see for example the paper by Roberts et al.)

Eq.10 of Tokovinin, PASP (2002):

$$S = \frac{I_{max}}{I_{tot}} \frac{4}{\pi} \left(\frac{\lambda_{CCD}}{D \Delta x} \right)^2$$

Quality of correction ? - 3

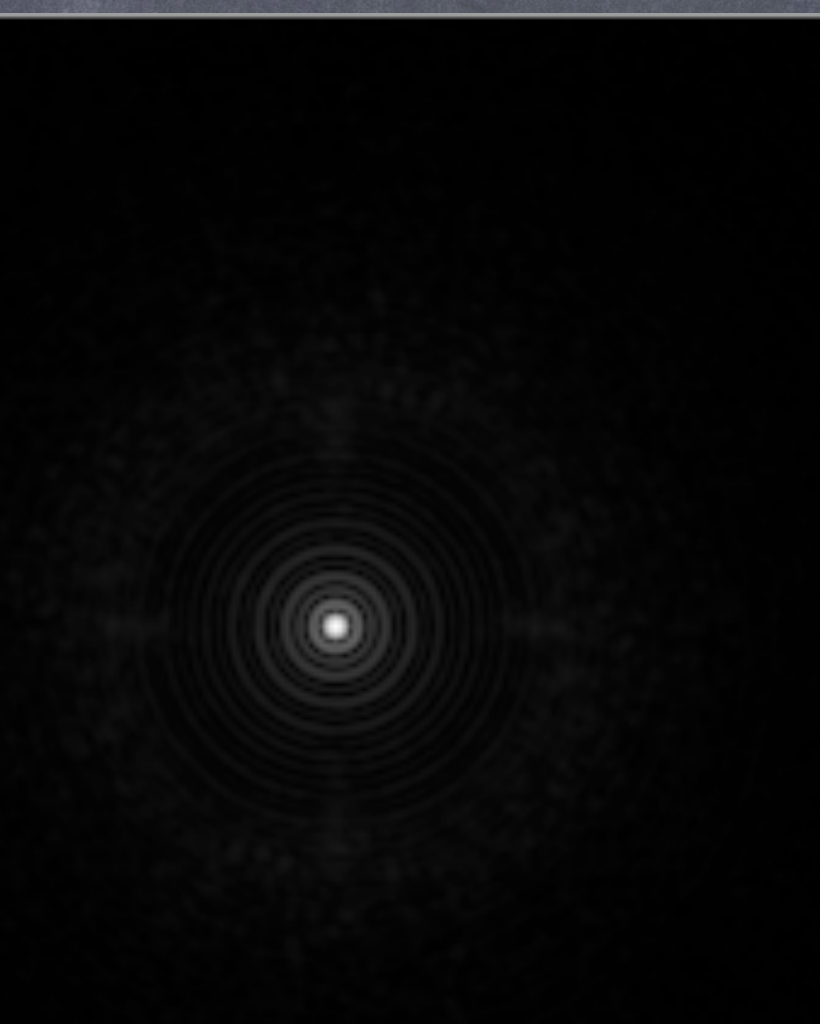


Quality of correction ? - 4



Strehl ~7%

This image shows a star with a very poor quality of correction, indicated by a Strehl ratio of approximately 7%. The star appears as a large, diffuse, and blurry white spot against a black background, with no distinct concentric rings or sharp features visible.



Strehl ~93%

This image shows a star with a high quality of correction, indicated by a Strehl ratio of approximately 93%. The star appears as a very sharp, bright white point with a series of well-defined, concentric diffraction rings surrounding it, indicating excellent optical performance.

Quality of correction ? - 5

Other parameters could be relevant:

- Full-width at half-maximum (FWHM) when the main question is resolution
- Encircled energy (EE) for spectroscopy
- Other non-direct criteria, such as:
 - . detection/SNR
 - . quality of image reconstruction

