Complex-space singularities of 2D Euler flow in
Lagrangian frame

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The main result reported here is that two-dimensional inviscid incompressible flow with
simple initial condition has not only Eulerian but also Lagrangian complex singularities. The
latter are closer to the real domain than the former.

To understand why we investigated this question, a few lines of context are needed. A solution
to the incompressible Euler equation, starting from entire initial data (e.g. trigonometric
polynomials), can be analytically continued to the complex space as long as it stays analytic
in the real space. As is known since the seventies, initial real analyticity for periodic solutions
is never lost in 2D [1]. The proven lower bound for the distance $\delta(t)$ to the real domain of
the nearest singularities decreases as a double exponential. However numerical simulations at
very high resolution (up to 8192\textsuperscript{2}) indicate that the decrease is much slower, close to a simple
exponential. The discrepancy seems related to the strong depletion of nonlinearity which is
systematically observed in 2D and 3D incompressible flow.

Recently it was shown by very precise simulations that in such 2D flow, the vorticity becomes
infinite at complex singularities [2, 3]. This implies that the corresponding (complex) fluid
particles must be located at infinity at $t = 0$. Could it be that, in Lagrangian coordinates, the
solution has no other singularity than at complex infinity? This would mean that the solution
is entire in Lagrangian coordinates. A simple counterexample is the AB flow $\psi = \sin x_1 \cos x_2$,
which is a steady solution to the 2D Euler equation in Eulerian coordinates. The trajectories
of fluid particles can then be integrated by elliptic functions and it was found in Ref. [4] that
fluid particles initially at suitable finite complex locations can escape to infinity at any real
positive time $t$.

What about flows which are not Eulerian steady-state solutions, such as the flow with the
initial condition $\psi = \cos x_1 + \cos 2x_2$? This question was investigated numerically using spectral
techniques with enough accuracy on high-order harmonics to allow the accurate determination
of complex singularities [5]. Direct application of spectral techniques in Lagrangian coordinates
is awkward. To calculate the solution in Lagrangian coordinates, we used the fact that the
inverse Lagrangian map $a(x, t)$ satisfies the advection equation $\partial_t a + u(x, t) \cdot \nabla a = 0$, where
$u$ is the Eulerian velocity field. This equation was solved by Eulerian spectral techniques
along with the 2D Eulerian equation for $u$. Then the Lagrangian map (or more precisely the
displacement $d \equiv x - a$) was calculated in Lagrangian coordinates. The inversion involved
the use of two uniform grids, a coarse Lagrangian and a finer Eulerian grid, together with a Newton iteration (details will be reported in the paper if the contribution is selected).

By composing the solution $u(x, t)$ with the Lagrangian map we obtained the Lagrangian velocity $u_L(a, t)$. From its 2D Fourier transform the Lagrangian width of the analyticity strip $\delta_L(t)$ was recovered in a standard way, by analyzing the high-wavenumber dependence of shell-summed Fourier amplitudes [2, 5].

The figure shows the wavenumber dependence of shell-summed amplitudes for the Eulerian and Lagrangian velocities at the same instant. They both exhibit exponential decay from which the Eulerian $\delta(t)$ and its Lagrangian counterpart $\delta_L(t)$ are measured. It is seen that the Lagrangian $\delta$ is always smaller than the Eulerian one. This could be a signature of the weaker depletion of nonlinearity in Lagrangian coordinates, compared to the Eulerian case (this observation was suggested to us by S. Orszag).

References


