Phase transitions and ensemble inequivalence in two-dimensional Euler flows

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We study analytically stable steady two-dimensional Euler flows in bounded domain, assuming a linear relationship between stream function and vorticity. Those states are minima of an energy-casimir functional, which is in this particular case the sum of a quadratic and a linear term in the vorticity field ω. We can either consider a microcanonical ensemble where energy $E[\omega]$ and circulation $\Gamma[\omega]$ are fixed, a canonical ensemble where the energy constraint is relaxed, or a grand canonical ensemble where both constraints are relaxed. Ensemble inequivalence appears when solutions of the variational problem with or without constraints are not the same.

In the context of the Robert-Sommeria-Miller statistical theory of two dimensional flows, Chavanis and Sommeria [1] reported and explained the existence of second order phase transitions in the microcanonical ensemble, if the (arbitrary) domain satisfies an explicit criteria. For example, there is a transition from a monopole to a dipole when the energy is increased in a rectangular domain whose aspect ratio is greater than 1.12.

We obtain the same criteria from a different approach. Our main contribution is to show the existence of ensemble inequivalence in this problem. We give (to our knowledge) the first example of second order azeotropy in a system with long range interactions: second order azeotropy had been described by Bouchet and Barre [2] in their classification of phase transitions for systems with symmetry, but had never been observe in physical systems;

it corresponds to the simultaneous appearance of two second order phase transitions.

We then generalize those results to multi-layers quasi-geostrophic equations. Geostrophic equilibrium reffers to the balance between Coriolis and pressure forces, which is particularly relevant to geophysical flows. In each layer, the potential vorticity field is advected by a non divergent two dimensional velocity field. The potential vorticity is the sum of the usual vorticity, a topographic term (a field $h(\mathbf{x})$), and a linear coupling term between streamfunctions of adjacent layers. In the case of a one layer model, taking into account variations of the Coriolis parameter with latitude, the linear relationship between potential vorticity and stream function leads to the well known Foffonof modes. The qualitative structure of Foffonof modes is a dipole with a weak westward flow at the middle of the domain, and intense recirculating eastward jets at the Northern and Southern boundaries. From our analytical results, we deduce qualitative structure of the solutions in more general cases, investigating the effect of different parameters : strength of the coupling between different layers, topography and shape of the domain. As in two-dimensional Euler equations, we report an ensemble inequivalence. We also give a method to predict the existence of second order phase transitions in the microcanonical ensemble : the domain must be such that it exists a zero-mean laplacian eigenmode, and the projection of the topography on the zero mean laplacian eigenmode associated with the smallest eigenvalue must be zero; in addition, the domain must satisfy a simple, explicit criteria as for Euler flows. We finally compare those solutions with real oceanic flows, to see if large scale coherent structures, as Gulf-Sream or Kuroshio, can be understood as equilibrium states.

References

[1] P.-H. Chavanis and J. Sommeria; Classification of self-organozed vortices in two-dimensional turbulence : the case of a bounded domain. *J. Fluid Mech.*, **314**, 1996

[2] F. Bouchet and J. Barre ; Classification of Phase Transitions and Ensemble Inequivalence, in Systems with Long Range Interactions. *J. Stat. Phys.*, **118**, 2005