Preferential concentration of inertial particles in turbulent flows

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EE250, Aussois, June 22, 2007
Particle laden flows

- Warm clouds
- Pyroclasts
- Plankton
- Protoplanetary disks
- Sprays
- Particle pollution

Finite-size and mass impurities advected by turbulent flow
Very heavy particles

- Spherical particles much smaller than the Kolmogorov scale $\eta$, much heavier than the fluid, feeling no gravity, evolving with moderate velocities: **one of the simplest model**

$$\frac{d^2 X}{dt^2} = -\frac{1}{\tau_s} \left[ \frac{dX}{dt} - u(X, t) \right] \implies 2 \text{ parameters:} \begin{cases} \text{St} = \tau_s/\tau_\eta \\ \text{Re} = U L/\nu \end{cases}$$

Prescribed velocity field
(random or solution to NS)

- **Dissipative dynamics** (even if $u(x, t)$ is incompressible)
Lagrangian averages correspond to an SRB measure that depends on the realization of the fluid velocity field.
Clustering of inertial particles

- Important for
  - the rates at which particles interact (collisions, chemical reactions, gravitation...)
  - the fluctuations in the concentration of a pollutant
  - the possible feedback of the particles on the fluid

Inertial-range clusters and voids

Multifractal at dissipative scales
Phenomenology of clustering

• Different mechanisms:
  
  Dissipative dynamics
  ⇒ *attractor*

Ejection from *eddies* by centrifugal forces

• **Idea:** find models to disentangle these two effects

Random flows uncorrelated in time
(isolate effect of dissipative dynamics)

*Lyapunov exponents*
*fractal dimensions*

Wilkinson & Mehlig; Falkovich & Co.

Simple model for both the flow and the dynamics able to reproduce the typical shape of the *mass distribution*
Mass distribution

- Coarse-grained density
  \[ \bar{\rho}_r(X, t) = \frac{1}{|B_r|} \int_{B_r(X(t))} d\mu_t \]

- Two asymptotics:
  \( r \to 0 \) (i.e. \( r \ll \eta \)): multifractal formalism
    large deviations for the ‘local dimension’ \( \ln \bar{\rho}_r / \ln r \)
  Question = dependence of the rate function \( S \) on the Stokes number \( St \)
  tools = Lyapunov exponents and their large deviations

  \( r \to \infty \) (i.e. \( r \gg \eta \)): how is uniformity recovered at large scales?
  use of the inertial-range properties of the flow
  Problem = not scale invariant anymore
  Question = how to account for a ‘scale-dependent inertia’?

\[ p(\bar{\rho}) \propto r^{S(\ln \bar{\rho} / \ln r)} \]
Small-scale clustering

- Linearized dynamics (tangent system)
  \[ \frac{d^2 R}{dt^2} = -\frac{1}{\tau_s} \left( \frac{dR}{dt} - \sigma(t)R \right) \]
  \[ \sigma_{ij}(t) = \partial_j u_i(X(t), t) \]
  \[ \lim_{t \to \infty} \frac{1}{t} \ln |R(t)| = \lambda \]
  \[ \text{Prob}(|R| < r) \propto r^{D_2} \]

Correlation dimension

Related to the radial distribution function
(see Sundaram & Collins 1997)

DNS (JB, Biferale, Cencini, Lanotte, Musacchio & Toschi, 2007)
Kraichnan flow

- Gaussian carrier flow with no time correlation
  Incompressible, homogeneous, isotropic

\[
\langle u_i(x, t) u_j(x', t') \rangle = [2D_0 \delta_{ij} - B_{ij}(x - x')] \delta(t - t')
\]

\[
B_{ij}(r) \sim D_1 r^{2h} [(d - 1 + 2h) \delta_{ij} - 2h r_i r_j / r^2]
\]

\[0 \leq h \leq 1 = \text{Hölder exponent of the flow}\]
Reduction of the dynamics

- Maps to a problem of Anderson localization
- Exponential separation
  Lyapunov exponent $\lambda = \langle X \rangle / \tau > 0$

Expansion in powers of the Stokes number
$= \text{diverging series} \Rightarrow \text{Borel resummation}$

Duncan, Mehlig, Östlund & Wilkinson (2005)
“Solvable” cases

- **One dimension** (Derevyanko et al. 2006)
  
  Potential: $U(X) = \frac{X^2}{2} + \frac{X^3}{3}$
  
  Constant flux solution: $p(X) \propto e^{-U(X)/S} \int_{-\infty}^{X} e^{U(X')/S} dX'$
  
  Lyapunov exponent: $\lambda = \frac{1}{2\tau} \left[-1 + c^{-1/2} \frac{\text{Ai}'(c)}{\text{Ai}(c)}\right]$ \(c \propto S^{-2}\)

- **Large-Stokes asymptotics** (Horvai nlin.CD/0511023)
  
  \[\lambda \propto D_1 S^{-2/3}\]
  
  + same scaling for FTLE

  $\mu(t) = \frac{1}{t} \ln[|\mathbf{R}(t)|/|\mathbf{R}(0)|]$  

  $p_t(\mu; S) \propto e^{-tD_1 S^{-2/3} h(S^{2/3} \mu/D_1)}$

(Bec, Cencini & Hillerbrand, 2007)
Inertial range distribution of mass

- Effective inertia decreases with $r$: scale invariance disappears
Small Stokes / Large box scaling

- The two limits $\tau_s \to 0$ and $r \to \infty$ are equivalent
- Naïve idea: **Local Stokes number** $St(r) = \frac{\tau_s}{\varepsilon^{-1/3} r^{2/3}}$
- Actually, scaling determined by the **increments of pressure**: Small inertia: Maxey’s approximation $\dot{X} \approx v(X, t)$ synthetic compressible flow: $v = u - \tau_s(\partial_t u + u \cdot \nabla u)$
- Relevant time scale for the time evolution of a blob of particles

$$\Gamma = \frac{1}{r^3} \int_{B_r} \nabla \cdot v \, d^3x \sim -\frac{\tau_s}{r} \Delta_r \nabla p$$

- Dimensional analysis: $\Delta_r \nabla p \sim \varepsilon^{2/3} r^{-1/3}$

Observed: scaling dominated by sweeping

$$\Delta_r \nabla p \sim U \varepsilon^{1/3} r^{-2/3}$$

so that $\Gamma \propto \tau_s r^{-5/3}$
Small Stokes / Large box scaling

- The density distribution depends only on $\Gamma \propto \tau_s r^{-5/3}$
Model for vortex ejection

Between two time steps: \( t \) and \( t + T \)

- **Flow divided in cells.**
  
  With a probability \( p \) the cells are rotating and eject particles to their non-rotating neighbors.

- Each cell contains a **continuous mass** \( m_j \) of particles. The mass ejected from the \( j \)th cell is at most \( \gamma m_j \).

(JB & Chéritée 2007)
One-cell mass distribution

After sufficiently large time:
the system reaches a **non-equilibrium steady state**.

PDF of $m_j$ very similar to that obtained in DNS (same tails)
Behavior of tails

- **Left tail**: algebraic \( p(m) \propto m^{\alpha(\gamma)} \) when \( m \ll 1 \)
  
  \[
  \begin{align*}
  M \text{ times} & \quad \text{Initial mass } m_j \approx 1 \\
  N \text{ times} & \quad \text{Final mass } m_j \approx (1 - \gamma)^N (1 - \gamma/2)^M
  \end{align*}
  \]

  \[
  \text{Prob} = [p(1-p)^2]^N [p(1-p)]^M \quad \Rightarrow \quad \alpha(\gamma, p)
  \]

- **Right tail**: super exponential

  \[
  m = \frac{1 - [1 - (1 - \gamma/2)^N]^M}{(1 - \gamma/2)^N}
  \]

  \[
  \text{Prob} = [p^2(1-p)]^{NM}
  \]

  \[
  \Rightarrow \quad p(m) \propto \exp(-C m \log m)
  \]
Summary / Open questions

- Two kinds of clustering
  - **Dissipative scales**: scale invariant (multifractal)
    relevant time scale \( \tau_\eta \)
  - **Inertial range**: scale invariance broken
    relevant time scale = acceleration (pressure gradient) \( \propto r^{5/3} \)

- What can be **analytically quantified**?
  - **Fractal dimensions / Lyapunov exponents**:
    Short-correlated flows: Wilkinson & Mehlig
    1D telegraph: Falkovich and coll.
  - **Inertial-range distributions** (cell ejection models)

- Is the 5/3 scaling a **finite Reynolds number effect**?