The Blowup Problem

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Outline:

1. Blowup

2. Criteria

3. Weak Solutions

4. Outlook

Euler Eqns

Eulerian

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ -\Delta p = \nabla \cdot (u \cdot \nabla u) \end{cases}$$

$\nabla \cdot u = 0$ = invariant constraint of incompressibility

Lagrangian

$$\begin{cases} a \mapsto X(a,t), \quad X(a,0) = a, \\ \partial_t^2 X + (\nabla_x p)(X,t) = 0, \\ -\Delta_x p = \\ \nabla_x \cdot \Big((\partial_t X \circ X^{-1}) \cdot \nabla_x (\partial_t X \circ X^{-1}) \Big) \end{cases}$$

$\det(\nabla_a X) = 1$ invariant constraint of incompressibility

Back-to-Labels $$\begin{cases} \partial_t A + u \cdot \nabla A = 0, \quad A(x,0) = x, \\ u = \mathbb{P}\left((\nabla A)^* u_0(A)\right) \end{cases}$$

Theorem 1 $u_0 \in C^s$, $s > 1$, $\nabla \cdot u_0 = 0$, $\nabla \times u_0 \in L^p$, $1 < p < \infty$. $\exists T > 0$, $A, u \in L^\infty([0,T], C^s)$.

BKM: Sufficient for regularity:

$$\int_0^T \|\omega\|_{L^\infty(dx)} dt < \infty$$

$$\omega = \nabla \times u.$$

Regularity = smooth solution on time interval $[0, T]$.

Necessary for applications

$$\int_0^T \|\nabla u\|_{L^\infty(dx)} dt < \infty$$

Vorticity evolution

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u$$

$$(\partial_t + u \cdot \nabla)\, |\omega| = \alpha |\omega|$$

$$\alpha = (\nabla u)\xi \cdot \xi = S\xi \cdot \xi,$$
$$\xi = \frac{\omega}{|\omega|},$$
$$S = \frac{1}{2}\left(\nabla u + (\nabla u)^*\right)$$

Sufficient for regularity:

$$\int_0^T \|\alpha\|_{L^\infty(dx)} dt < \infty$$

$$S_{ij}(x,t) = \frac{3}{8\pi} P.V. \int_{\mathbb{R}^3} \left(\epsilon_{ipk} \widehat{y}_j + \epsilon_{jpk} \widehat{y}_i \right) \widehat{y}_p \omega_k (x-y) \frac{dy}{|y|^3}$$

$$\alpha(x,t) = \frac{3}{4\pi} P.V. \int_{\mathbb{R}^3} D(\widehat{y}, \xi(x-y,t), \xi(x,t)) |\omega(x-y,t)| \frac{dy}{|y|^3}$$

$$D(e_1, e_2, e_3) = (e_1 \cdot e_3) \det(e_1, e_2, e_3)$$

$$|D(\widehat{y}, \xi(x-y,t), \xi(x,t))| \le |\xi(x-y) \times \xi(x)| = |\sin\phi|$$

At worst locally osculating anti-parallel vortex lines = local sine-Lipschitz

$$|\xi(x-y,t) \times \xi(x,t)| \le C_a(t)|y|, \quad \text{for } |y| \le r(t)$$

At worst locally osculating parallel vortex lines = local Lispchitz ξ.

$$|\xi(x-y,t) - \xi(x,t)| \le C_p(t)|y|, \quad \text{for } |y| \le r(t)$$

Clearly, local Lipschitz implies local sine-Lipschitz but not vice-versa.

$$|\xi(x-y) - \xi(x)| = 2\left|\sin\left(\frac{\phi}{2}\right)\right|$$

Soft cut-off at r: inner stretching factor

$$\alpha^r(x,t) = \frac{3}{4\pi} P.V. \int \chi\left(\frac{y}{r}\right) D(\widehat{y}, \xi(x-y,t)\ , \xi(x,t))|\omega(x-y,t)|\frac{dy}{|y|^3}$$

Outer rate of strain:

$$(S_\rho(x,t))_{ij} =$$
$$\frac{3}{8\pi} P.V. \int \left(1 - \chi\left(\frac{y}{\rho}\right)\right)\left(\epsilon_{ipk}\widehat{y}_j + \epsilon_{jpk}\widehat{y}_i\right)\widehat{y}_p \omega_k(x-y)\frac{dy}{|y|^3}$$

$$S_r^\rho(x,t) = \int \left(\chi\left(\frac{y}{\rho}\right) - \chi\left(\frac{y}{r}\right)\right)\dots$$

$$\boxed{\alpha(x,t) = \alpha^r(x,t) + \alpha_r^\rho(x,t) + \alpha_\rho(x,t)}$$

$$r \le r(t)$$

$$|\alpha^r(x,t)| \le r C_a(t) \sup_{|x-z|\le r} |\omega(z,t)|$$

$$U(x,t) = \sup_{|x-z|\le \rho} |u(z,t)|$$

$$|\alpha_r^\rho(x,t)| \le c \frac{U(x,t)}{r}$$

$$|\alpha_\rho(x,t)| \le c\rho^{-\frac{3}{2}} \|u_0\|_{L^2}$$

Method of CFM, Sufficient for regularity:

$$\begin{cases} \xi \quad \text{locally sine-Lipschitz} \\ u \quad \text{locally bounded} \\ \int_0^T \inf_{r \le r(t)} \left\{ \frac{U(t)}{r} + rC_a(t) \|\omega(t)\|_{L^\infty} \right\} dt < \infty \end{cases}$$

Example:

$r(t) \sim (T-t)^a$, $U(t) \sim (T-t)^{-b}$, $C_a(t) \sim (T-t)^{-c}$, $\|\omega\|_{L^\infty} \sim (T-t)^{-1}$.

If $1-b+c > 2a$ then the condition for absence of singularity is $b+c<1$.

If $1-b+c < 2a$ then the condition is $a+b<1$ and $0 < a-c$.

Moreover...

$$D(\widehat{y},\omega(x-y),\xi(x)) = (\widehat{y}\cdot\xi(x))\times$$
$$\{[(\xi(x)\cdot\nabla_y)u(x-y)]\cdot\widehat{y} - (\widehat{y}\cdot\nabla_y)(\xi(x)\cdot u(x-y))\}$$

If $|\xi(x,t)\cdot u(z,t)| \leq U_{par}$ is locally bounded and if the vortical region becomes thin, then

$$|\alpha_r^{\rho}(x,t)| \leq c\frac{U_{par}}{r} + c\frac{U}{r}\left(\frac{w}{r}\right)^g$$

where $w(t)$ is the width of the vortical region, $g = 1$ for sheet-like structures, $g = 2$ for tube-like structures. If $w \sim (T-t)^d$, $d > a$: gain of the $\min\{g(d-a)\ , 1-c\}$ over previous.

$$\omega = C(\nabla A, \omega_0(A))$$

From BKM: Sufficient for regularity

$$\int_0^T \|\nabla A\|^2_{L^\infty(dx)} dt < \infty$$

Chae: No single scale self similar blowup.

Fefferman and Cordoba; No squirt blowup without infinite velocity.

Deng, Hou and Yu: local analysis along vortex line and extension of CFM adapted to specific computations.

Let

$$\Pi(x,t) = \left(\frac{\partial^2 p}{\partial x_i \, \partial x_j} \right)$$

and consider

$$Q(t) = \{x \mid \Pi(x,t) > 0\}$$

the region where Π is positive definite. (Note that nondegenerate local minima of $p(x,t)$ are in $Q(t)$.)

Sufficient for blowup:

$$\begin{cases} \exists a \in Q(0), \ \text{ such that } X(a,t) \in Q(t), \ \forall t \in [0,T] \\ \omega_0(a) = 0, \\ T\rho(S_0)(a) > 3 \quad \text{where } \rho(S_0) = \ \text{ spectral radius} \end{cases}$$

Idea used to prove blowup for “distorted Euler equations” (’86).

$$D_t S + S^2 + \Pi - \frac{|\omega|^2}{4} P_\omega^\perp = 0$$

The proof is by contradiction. By assumption $\exists \phi_0$ so that

$$\begin{cases} \int_{\mathbb{R}^3} |\phi_0(a)|^2 da = 1, \\ \int_{\mathbb{R}^3} S_0(a)\phi_0(a) \cdot \phi_0(a) da < 0, \\ T \left| \int_{\mathbb{R}^3} S_0(a)\phi_0(a) \cdot \phi_0(a) da \right| > 1 \end{cases}$$

and also, if we solve

$$D_t \phi = 0, \quad \phi(a, 0) = \phi_0(a)$$

then

$$supp \phi(t) \subset Q(t) \quad \text{holds, for } 0 \le t \le T.$$

We take

$$y(t) = \int S(x,t)\phi(x,t) \cdot \phi(x,t) dx$$

This blows up before T:

$$\frac{d}{dt}y + y^2 \leq 0$$

because

$$|\omega(x,t)|^2|\phi(x,t)| = 0,$$

$$\int_{\mathbb{R}^3} |\phi(x,t)|^2 dx = 1$$

and Schwartz

$$\int_{\mathbb{R}^3} |S\phi|^2 dx \geq y^2(t). \quad \square$$

Work by Ohkitani, (and Klainerman '84, unpublished) uses Π and the equation for S to compute the second material derivative of $|\omega|^2$.

Weak Solutions

Desirable: locally square integrable, evolutionary weak solutions obtained as limits of good approximate solutions u^ϵ. Needed: weak continuity of approximations in L^2. (Weak continuity is stronger than strong continuity).

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} \{(u^\epsilon \otimes u^\epsilon) : \nabla\phi\}\, dx = \int_{\mathbb{R}^3} \{(u \otimes u) : \nabla\phi\}\, dx$$

for all smooth divergence–free ϕ, when

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} (u^\epsilon \cdot \phi)\, dx = \int_{\mathbb{R}^3} (u \cdot \phi)\, dx$$

holds for all ϕ.

Known for surface QG (Resnick, '95), not for Euler.

QG

$$\partial_t\theta + u\cdot\nabla\theta = 0, \quad u = R^{\perp}\theta.$$

Active scalar (*Geometric statistics in turbulence*, SIAM Review '94: all a equations.) For periodic $\theta = \sum_{j\in\mathbb{Z}^2}\widehat{\theta}(j)e^{i(j\cdot x)}$ infinite ODE:

$$\frac{d}{dt}\widehat{\theta}(l) = \sum_{j+k=l}\left(j^{\perp}\cdot k\right)|j|^{-1}\widehat{\theta}(j)\widehat{\theta}(k)$$

Using the antisymmetry:

$$\frac{d}{dt}\widehat{\theta}(l) = \sum_{j+k=l}\gamma^{l}_{j,k}\widehat{\theta}(j)\widehat{\theta}(k)$$
$$\gamma^{l}_{j,k} = \frac{1}{2}(j^{\perp}\cdot k)\left(\frac{1}{|j|} - \frac{1}{|k|}\right)$$

$$\left|\gamma_{j,k}^{l}\right| \le \frac{|l|^2}{\max\{|j|\,,|k|\}}$$

Consequently

$$\|(-\Delta)^{-1}\left[B(\theta_1,\theta_1) - B(\theta_2,\theta_2)\right]\|_w \le$$
$$C\left\{\|\theta_1-\theta_2\|_w\left(1+\log_+\|\theta_1-\theta_2\|_w\right)\right\}\left(\|\theta_1\|_{L^2}+\|\theta_2\|_{L^2}\right)$$

with $\|f\|_w = \sup_{j\in\mathbb{Z}^2}\left|\widehat{f}(j)\right|$. Quasi-Lipschitz, with loss of two derivatives. Loss of derivatives does not impede existence theory, but prevents a proof of uniqueness. Regularity and uniqueness: with critical dissipation ($|k|\widehat{\theta}(k)$): Cordoba-Wu-C (small data), Kiselev-Nazarov-Volberg and Caffarelli-Vasseur, all data. For supercritical dissipation there is a gap in passing from L^∞ to C^s, no gap in passing from L^2 to L^∞, nor from C^s to C^∞ (Wu-C, Caffarelli-Vasseur).

Littlewood-Paley decomposition.

$$u = \sum_{j=-1}^{\infty} \Delta_j u$$

$$\text{supp}\,\mathcal{F}(\Delta_j(u)) \subset 2^j[\tfrac{1}{2}, \tfrac{5}{4}]$$
$$\Delta_j \Delta_k \neq 0 \Rightarrow |j-k| \leq 1,$$
$$\left(\Delta_j + \Delta_{j+1} + \Delta_{j+2}\right) \Delta_{j+1} = \Delta_{j+1}$$
$$\Delta_j \left(S_{k-2}(u) \Delta_k(v)\right) \neq 0 \Rightarrow k \in [j-2, j+2]$$
$$S_k(u) = \textstyle\sum_{q=-1}^{k} \Delta_q.$$

$$\Delta_j = \Psi_j(D) = \Psi_0(2^{-j}D), \quad \Delta_{-1}u = \Phi_{-1}(D)u.$$

Φ_{-1}: radial, nonincreasing, C^∞

$$\begin{cases} \Phi_{-1} = 1, \ 0 \le r \le a \\ \Phi_{-1} = 0, \ r \ge b \\ 0 < a < b < 1 \end{cases}$$

$$\Psi_0(r) = \Phi_{-1}(r/2) - \Phi_{-1}(r), \quad \Psi_j(r) = \Psi_0(2^{-j}r).$$

$$(\Psi(D)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi)}\Psi(\xi)\widehat{u}(\xi)d\xi$$

$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i(x\cdot\xi)}u(x)dx.$ $a < b < \frac{4}{3}a$ (e.g. $a = 1/2, \ b = 5/8$)

Inhomogeneous Besov space

$$\|u\|_{B^s_{p,q}} = \left\| \left\{ 2^{sj} \|u_j\|_{L^p} \right\}_j \right\|_{\ell^q(\mathbb{N})} .$$

The space $B^s_{p,c(\mathbb{N})}$ is the subspace of $B^s_{p,\infty}$ formed with functions such that

$$\lim_{j\to\infty} 2^{sj} \|u_j\|_{L^p} = 0.$$

The Littlewood-Paley energy flux is

$$\Pi_N := \int_{\mathbb{R}^3} \text{Trace } \left[S_N(u \otimes u) \nabla S_N(u) \right] dx.$$

This is the (formal) time derivative

$$\Pi_N = \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} |S_N(u(t))|^2 \, dx$$

of a the energy contained in $S_N(u)$ when u solves the Euler equation.

Onsager Conjecture

$$u \in C^{\frac{1}{3}} \Leftrightarrow \frac{dE}{dt} = 0$$

Eyink, C-E-Titi, Duchon-Robert

Theorem 2 *(Cheskidov-C-Friedlander-Shvydkoy) Weak solutions*

$$u \in L^3([0, T], B^{1/3}_{3,c(\mathbb{N})})$$

of the Euler equations conserve energy. There exist functions in $B^{1/3}_{3,\infty}$ *that are divergence-free and obey* $\liminf_{N\to\infty} |\Pi_N| > 0$.

Related results, also helicity. See also work of Chae. In two dimensions, infinite time, damped and driven NS: absence of anomalous dissipation of enstrophy. (Ramos-C, poster.)

Euler weak solutions: main difficulty

$$B(u,v) = \mathbb{P}(u \cdot \nabla v) = \Lambda \mathbb{H}(u \otimes v)$$

where

$$[\mathbb{H}(u \otimes v)]_i = R_j(u_j v_i) + R_i(R_k R_l(u_k v_l)),$$

$\mathbb{P}$ is the Leray-Hodge projector, $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the Zygmund operator and $R_k = \partial_k \Lambda^{-1}$ are Riesz transforms.

$$\Delta_q(B(u,v)) = C_q(u,v) + I_q(u,v)$$

$$C_q(u,v) = \sum_{p\ge q-2,\ |p-p'|\le 2} \Delta_q(\Lambda\mathbb{H}(\Delta_p u, \Delta_{p'} v))$$

$$I_q(u,v) = \sum_{j=-2}^{2} \left[\Delta_q \Lambda \mathbb{H}(S_{q+j-2}u, \Delta_{q+j}v) + \Delta_q \Lambda \mathbb{H}(S_{q+j-2}v, \Delta_{q+j}u)\,\right]$$

For L^2 weak solutions it would be desirable to have a bound of the type

$$\|\Lambda^{-M}\left(B(u_1,u_1) - B(u_2,u_2)\right)\|_w \le C\|u_1 - u_2\|_w^a \left[\|u_1\|_{L^2} + \|u_2\|_{L^2}\|\right]^{2-a}$$

with $a > 0$ and $\|f\|_w$ a weak enough norm so that weak convergence in L^2 implies convergence in the w norm, (e.g $B^{-s}_{\infty,\infty},\ s > 3/2$) and M as large as needed. This is true for $I(u,v)$ but not for $C(u,v)$. For weak solutions in $B^{\frac{1}{3}}_{3,q}$, $C(u,v)$ is good and $I(u,v)$ is bad.

Outlook

●No blow up. How to prove it? Geometric estimates (solution determines space).

●Weak solutions in right spaces? More nonlinear structure is needed.

●Anomalous dissipation? Maybe for special stationary statistical Euler solutions.