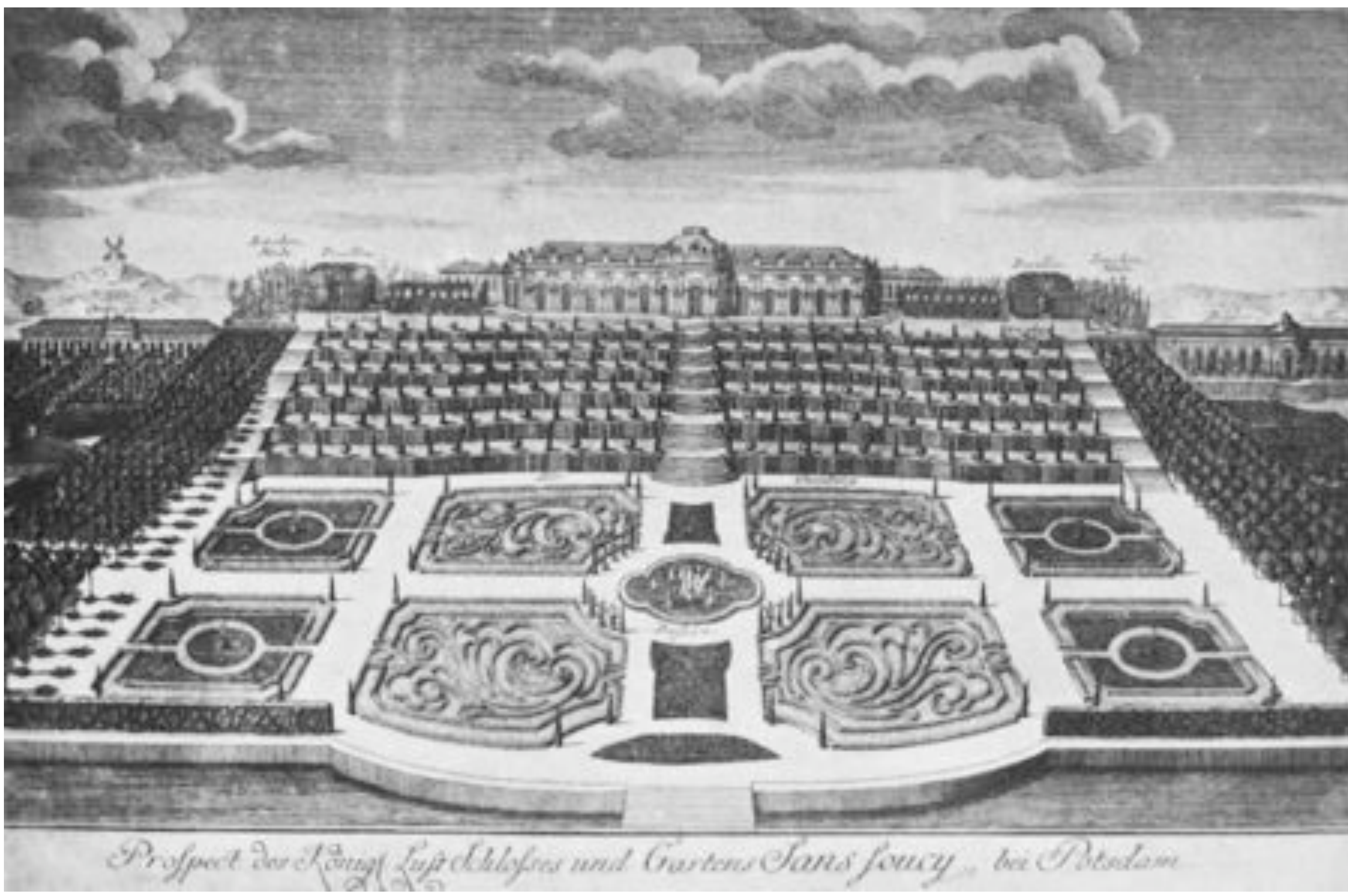
Water-art problems at Sanssouci

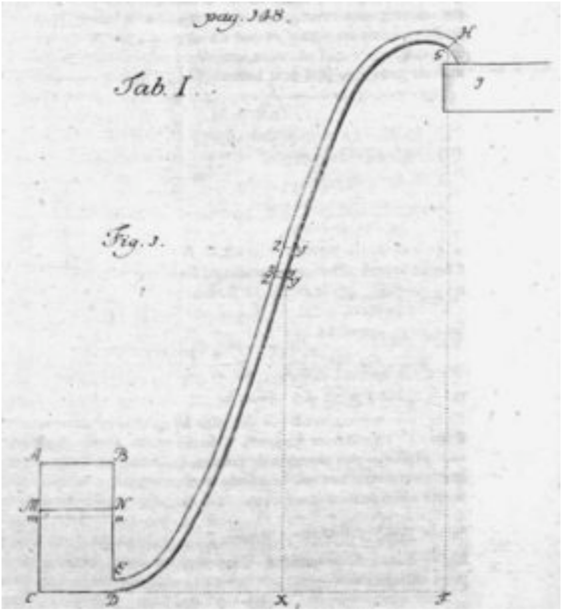
Euler's involvement in practical hydrodynamics on the eve of ideal flow theory

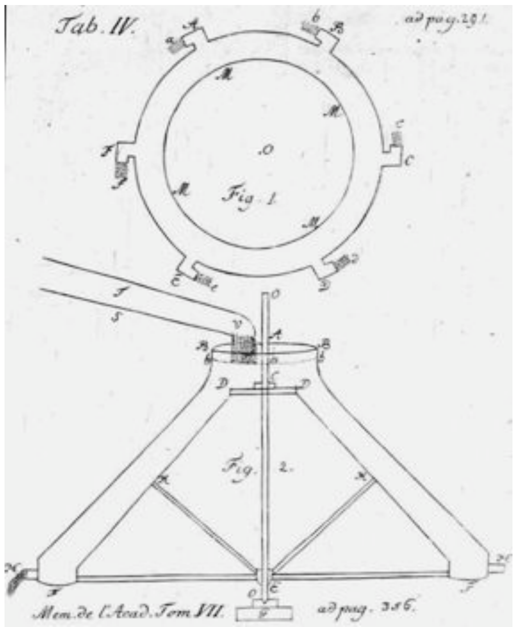


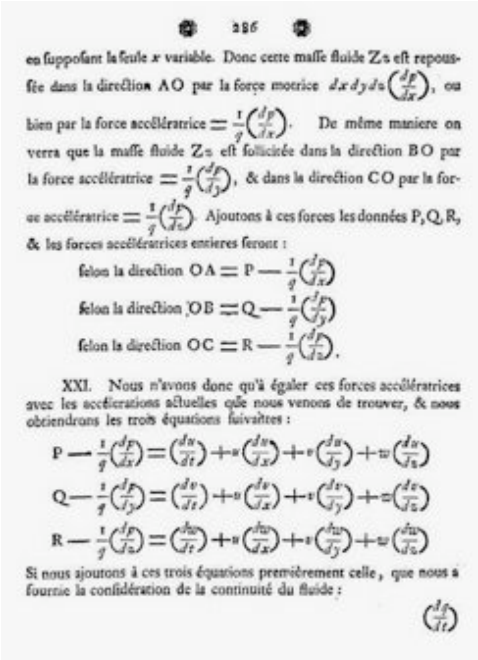
Thank you for inviting me to this conference. I do not feel as an expert on Euler or on 18th century hydrodynamics, but I became interested in Euler's alleged failure for the fountains at Sanssouci, as an example of the proverbial gap between theory and practice. So I studied this case more closely. The results of this study are the subject of this lecture.

Euler - practical academician

1741/1749	Scientia Navalis
1744/1745	Neue Grundsätze der Artillerie
1749/1752	**Sur le mouvement de l'eau par des tuyaux de conduite**
1750/1760	Recherches sur le mouvement des rivieres
1751/1752	Recherches sur l'effet d'un machine hydraulique
1755/1757	Principes generaux du mouvement des fluides







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en supposant la seule x variable. Donc cette masse fluide Z est repoussée dans la direction AO par la force motrice $dx\,dy\,dz\left(\frac{dp}{dx}\right)$, ou bien par la force accélératrice $= \frac{1}{q}\left(\frac{dp}{dx}\right)$. De même maniere on verra que la masse fluide Z est sollicitée dans la direction BO par la force accélératrice $= \frac{1}{q}\left(\frac{dp}{dy}\right)$, & dans la direction CO par la force accélératrice $= \frac{1}{q}\left(\frac{dp}{dz}\right)$. Ajoutons à ces forces les données P, Q, R, & les forces accélératrices entieres seront :

selon la direction OA $= P - \frac{1}{q}\left(\frac{dp}{dx}\right)$

selon la direction OB $= Q - \frac{1}{q}\left(\frac{dp}{dy}\right)$

selon la direction OC $= R - \frac{1}{q}\left(\frac{dp}{dz}\right)$.

XXI. Nous n'avons donc qu'à égaler ces forces accélératrices avec les accélerations actuelles que nous venons de trouver, & nous obtiendrons les trois équations suivantes :

$$P - \frac{1}{q}\left(\frac{dp}{dx}\right) = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right)$$

$$Q - \frac{1}{q}\left(\frac{dp}{dy}\right) = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right)$$

$$R - \frac{1}{q}\left(\frac{dp}{dz}\right) = \left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right)$$

Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la considération de la continuité du fluide :

$\left(\frac{dq}{dt}\right)$

As a rule, the agendas of the academies in the 18th century were much more practical than it appears in retrospect, when we speak of “academic” science and associate with it “pure” science. The Berlin academy, with Euler as director of the mathematical class, is no exception. These are only some of Euler’s more practical works during his Berlin years which had something to do with hdrodynamics:

- he was involved with naval architecture from early on, and published a famous treatise;
- the same is true for ballistics, which included the problem of fluid resistance;
- he did really important work on the hydraulics of pipeflow - motivated by the Sanssouci project;
- he was involved in a study of channel flow, for making a canal navigable;
- he did pioneering work on hydraulic machinery, from which originated “Euler’s turbine theory”

So, when he formulated his general equations for fluid motion, this was based on at least a decade of problem solving in practical hydrodynamics.

„... second rate as a physicist..."

„At times Mr. Euler appeared only to enjoy the pleasures of calculation... Mr. Euler the Metaphysician or even the Physicist was not as great as the Geometer... he only wished to exhibit the power of his art." (Marquis de Condorcet 1783)

„The physical universe was an occasion for mathematics to Euler, scarcely a thing of much interest in itself; and if the universe failed to fit his analysis it was the universe which was in error." (E. T. Bell 1937)

„The mathematical genius Euler was second rate as a physicist." (A. Hermann 1991)

"When Euler applied his equations to design a fountain for Frederick the Great of Prussia, it failed to work...Unfortunately, he omitted the effects of friction, with embarrassing practical consequences." (S. Perkovitz 1999)

Nevertheless, according to a widespread slander, Euler's ability with practical matters was limited. These are some examples how Euler entered popular accounts.

The slander began already immediately after Euler's death with Condorcet's eulogy, who praised him highly as a mathematical genius - but said this about his practical abilities: „At times..."

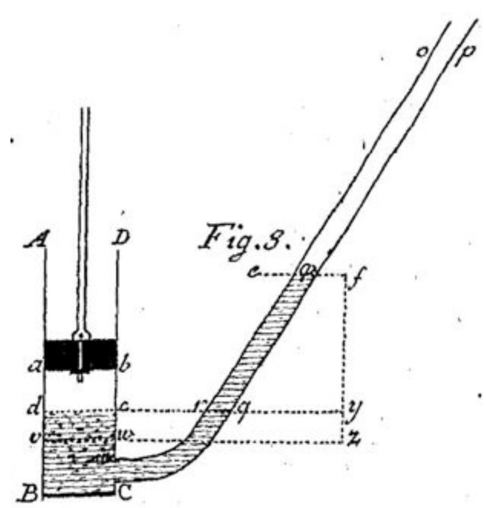
This became the source of future portrayals, like E. T. Bell's famous account *Men of Mathematics*: „The physical universe..."

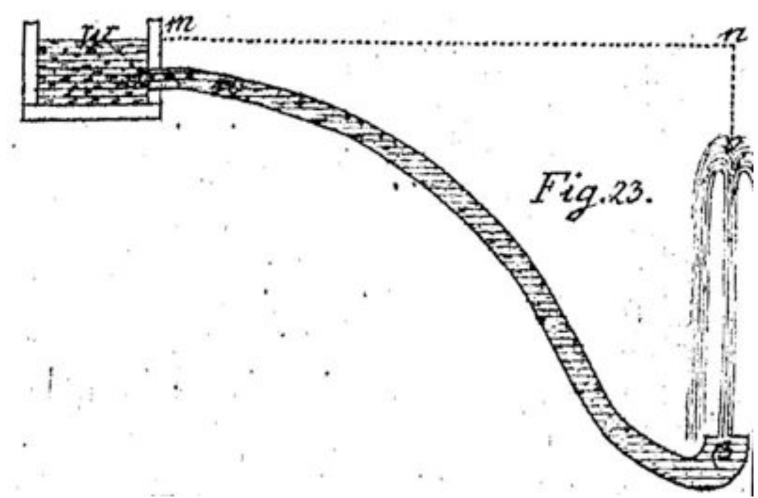
A historian of physics simply concluded: „The mathematical genius..."

And a physicist suggested that it was Euler's focus on ideal fluid theory which led him astray with practice: „When Euler applied...."

Frederick‘s slander

Je voulus faire un jet-d'eau en mon Jardin; le Ciclope Euler calcula l'éffort des roües, pour faire monter l'eau dans un bassin d'où elle devoit retomber par des Canaux, afin de jaillir à Sans-Souci. Mon Moulin a été éxécuté géométriquement, et il n'a pu élever une goute d'eau à Cinquante pas du Bassin. Vanité des Vanités; Vanité de la géométrie.” (Frederick II to Voltaire, 25 January 1778)



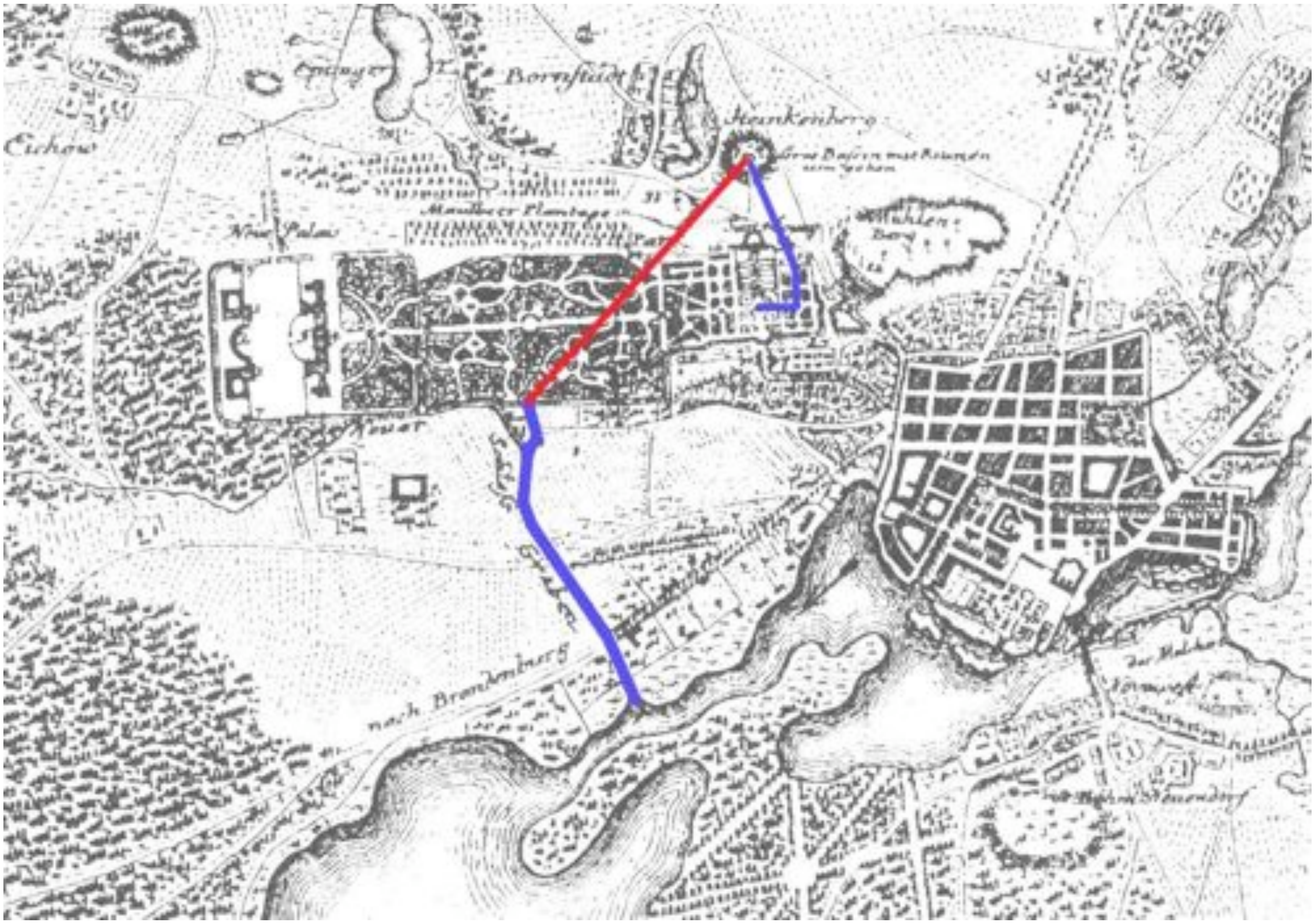


The Sanssouci affair seems to have started this view of Euler. This is what Frederick the Great wrote in a letter to Voltaire:

“I wanted to make a fountain jet in my Garden; the Cyclop Euler calculated the effort of the wheels for raising the water to a basin, from where it should fall down through canals, in order to form a fountain jet at Sans-Souci. My mill was constructed mathematically, and it could not raise one drop of water to a distance of fifty feet from the basin. Vanity of Vanities! Vanity of mathematics.”

The problem was how to raise water. Pumps, driven by a windmill or horses, pressed water in a pipeline up into an elevated reservoir. From there it would feed the fountains. Something went wrong with this design. And, indeed, the project failed. Only in the 19th century the water art project for Sanssouci was successfully completed.

What happened at Sanssouci?



What was the problem at Sanssouci?

Here you see the plan upon which the design of the water art was based:

Water from the river Havel was guided to a pump station at one end of the Park. From there it was pumped in a pipeline which ended at the elevated reservoir. Other pipes led from the reservoir to the fountains close to the castle. The pipeline to the reservoir (red) turned out to be the source of the problems. It was about 1 km long; the height difference between pump and reservoir was about 50 meters.

Heinrich Ludewig Manger's
Königl. Preuß. Ober-Hof-Baurath und Garteninspectors

Baugeschichte

von

Potsdam,

besonders
unter der Regierung

König Friedrichs des Zweiten.

Erster Band,
welcher die Baugeschichte von den ältesten Zeiten bis 1762 enthält.

Berlin und Stettin,
bei Friedrich Nicolai, 1789.

vier bleyerne Grouppen, sechsfüßiger Proportion; Giese goß überdieß die metallenen Cylinder, oder Stiefel zur Mühle und verschiedene Arbeiten von Bley, weil er im Gießen sehr geschickt war. 1748

<u>Die hölzernen gebohrten Röhren mit den eisernen Schraubenringen waren nun endlich auch verlegt, erfuhren aber ebenfalls gar bald das Schicksal der vorigen aus verschiedenen Stäben zusammengesetzten, nämlich sie konnten den Druck des Wassers nicht aushalten und zersprangen.</u> Dem Könige mußte also vorgestellet werden, daß auf keine Weise das Wasser anders herauf in den Sammelkasten zu bringen möglich wäre, als wenn solches vermittelst guter gegossenen eisernen, oder bleyernen Röhren geschähe, wozu denn dergleichen von Cassel und vom Harze vorgeschlagen wurden.

Es konnte nicht anders seyn, als daß der König wegen der durch zweierley hölzernen Röhren verwendeten, gleichsam weggeworfenen Summen, sehr unwillig ward, denn er verlangte durchaus und mit Recht, daß diejenigen, welche etwas angäben, vorher von dem Effekte durch Erfahrungen sicher seyn müßten. Er äusserte indessen den gerechten Unwillen nicht auf eine strenge Art, sondern da er von neuem zu eisernen Röhren Gelder assignirt hatte: so ließ er ein Paar Esel in Lebensgröße mit Oelfarbe auf Leinwand mahlen, mit Rähmen umgeben, und die Unterschrift beifügen:

Hollaandse Fontaenen-Maacker.

Diese recht schön, und gänzlich nach der Natur getroffenen Thiere, sollten mit Wasserfarbe, etwas anders vorstellend übermahlt, und an ein

Fortunately we have a very detailed account about the history of constructions at Potsdam, including the Sanssouci project, under the reign of Frederick the Great. It was written by the King's last architect and published in 1789. The Sanssouci water art project is described on about ten pages. To make along story short: It began in 1748, experienced one failure after another, was interrupted in 1756 because of the Seven-Years-War, resumed in 1763 without success, and finally abandoned a few years later. The first problem was that the tubes used for the pipeline were made from wood; they burst before the water reached far enough to the reservoir. Later, when metal tubes were used, they were not dimensioned so that water in sufficient quantity could be pumped through.

The account lists many names of people who contributed to the yearlong bungling at Sanssouci - but Euler is not mentioned.

Euler‘s involvement

« ... j'ai l'honneur de Vous marquer que j'expédiai hier au Roy mes recherches sur la lotterie projettée, et que j'espère de venir à bout en quelques jours de celles sur la machine hydraulique ... » (Euler to Maupertuis, 18 September 1749)

« Je prend la liberté de vous addresser mes recherches sur la Machine Hydraulique de Sans Soucy ... je crains fort qu'il s'en faudra beaucoup qu'elle monte à la hauteur que Le Roy souhaite... » (Euler to Maupertuis, 21 September 1749)

« Comme Sa Majesté le Roy de Prusse, Notre très gracieux Souverain, a reçu les calculs que le professeur Euler Lui a adressé au sujet de la Machine de Sans-Souci et qu'Elle en est fort contente, Sa Majesté veut bien lui témoigner tout le gré ... » (Frederick to Euler, 27 September 1749)

« ... en cas que l'expérience de Mariotte ne fût pas juste, ou gâtée par une faute d'impression, je ne saurois rien déterminer sur l'épaisseur des tuyaux dans le cas dont il s'agit, à moins qu'on ne fît de nouveau des expériences sur la force que des tuyaux de plomb sont capables de soutenir. Car on risqueroit trop si l'on vouloit confier au seul hazard la détermination de l'épaisseur des tuyaux ... » (Euler to Maupertuis, 30 September 1749)

But Euler became indeed involved in the Sanssouci project. These are quotes from his correspondence with Maupertuis, the President of the Berlin academy, and with the King himself in September 1749:

Euler‘s letter to Maupertuis on September, 30th, is particularly interesting: He recommends to undertake experiments for determining the appropriate wall thickness of lead tubes for the pipeline.

Euler‘s warning

« Car sur le pied qu'elles se trouvent actuellement, il est bien certain, qu'on n'éleveroit jamais une goutte d'eau jusqu'au réservoir, et toute la force ne seroit employée qu'à la destruction de la machine et des tuyaux. » (Euler to Frederick, 17 October 1749)

« La véritable cause de ce fâcheux accident consistoit uniquement en ce que la capacité des pompes étoit trop grande, et à moins qu'on ne la diminue très considérablement, ou en diminuant leur diamètre ou leur hauteur, ou le nombre des jeux qui repond à un tour de moulin, la machine ne sera pas en état de fournir une seule goûte d'eau dans le réservoir. » (Euler to Maupertuis, 21 October 1749)

« J'ai reçu votre lettre du 17^{e} de ce mois, contenant les remarques, que vous avez fait sur vos calculs sur les pompes et les tuyaux de la Machine de Sans-Souci. Elles M'ont été fort agréables, et Je vous suis bien obligé de la peine que vous en avez pris. » (Frederick to Euler, 21 October 1749)

In this correspondence Euler explicitely called attention to the appropriate dimensions of the pipes with regard to the power of the pumps. He refered to the case when the wooden pipes burst for the first time. The King thanked Euler - but did not draw consequences from Euler‘s advice. So the bungling proceeded unabatedly. There is no evidence that Euler was involved furthermore.

In view of this short involvement, for about a month in autumn 1749, and the yearlong history of further bungling without taking Euler‘s advice into account, it is not astonishing that Euler‘s name is not mentioned in the book about the history of constructions under the reign of Frederick the Great. So Frederick‘s slander in his letter to Voltaire, written many years after Euler‘s short-termed involvement, is the only document upon which all subsequent interpretations and slanders were based.

Euler‘s Sanssouci memoirs

Sur le mouvement de l'eau par des tuyaux de conduite (E 206)
(presented to the Berlin Academy on October 23, 1749; published in *Mémoires de l'académie des sciences de Berlin* 8, (1752) 1754, pp. 111-148.

Discussion plus particuliere de diverses manieres d'elever de l'eau par le moyen des pompes avec le plus grand avantage (E 207)
(presented to the Berlin Academy on November 20, 1749; published in *Mémoires de l'académie des sciences de Berlin* 8, (1752) 1754, pp. 149-184.)

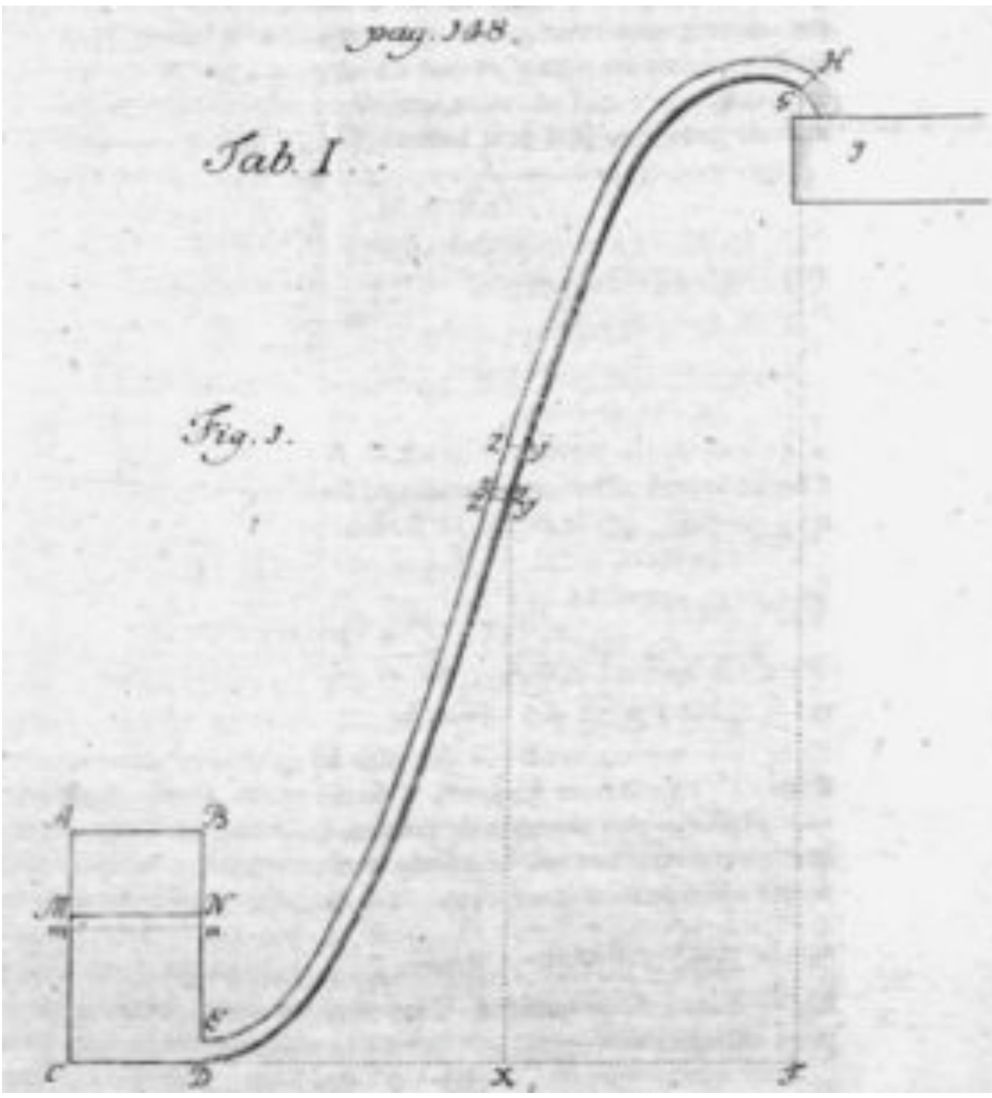
Maximes pour arranger le plus avantageusement les machines destinées è élever de l'eau par le moyen des pompes (E 208)
(presented to the Berlin Academy on February 5, 1750; published in *Mémoires de l'académie des sciences de Berlin* 8, (1752) 1754, pp. 185-232.)

It would be another story to explain why Frederick perverted Euler‘s warning about further bungling at Sanssouci in its opposite, and even blamed him for the failure.

Here I rather content myself with the objective results of Euler‘s involvement: He wrote three memoirs for the Berlin academy in which he presented a detailed theory on pipeflow and its practical consequences for water raising machinery.

For the rest of my lecture I focus only on the first of these memoirs, because it deserves also some interest for Euler‘s further theorizing on fluid motion.

Euler's pipeflow theory (E 206)



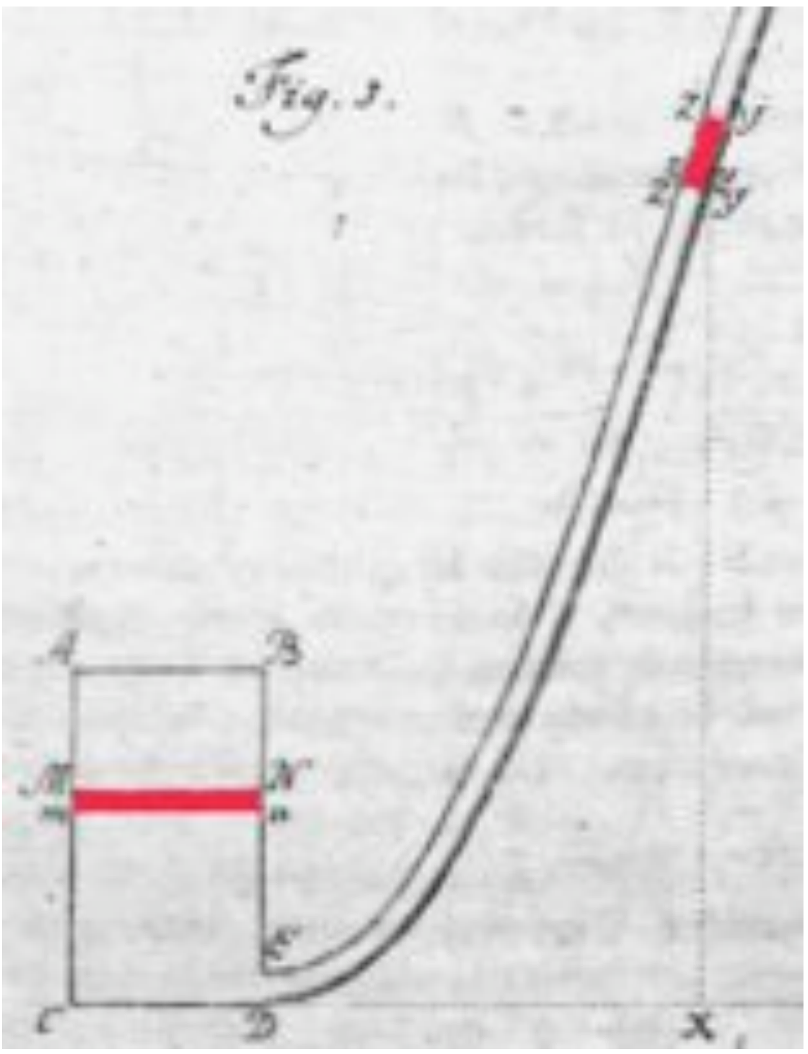
1. Equation of Motion
2. Integration
3. Applications

Here you see the Sanssouci water raising problem as posed by Euler in his pipeflow memoir.

A piston pushes water down a pump cylinder into a pipeline which ends at an elevated reservoir.

Euler first formulates the equation of motion for this problem; he then integrates it; and finally derives some formulae which are important for practical applications.

Equation of motion



1. Calculate acceleration at *YY'* if piston moves down *Mm*

2. Consider balance of forces for the slice *YZzy*

De ces deux forces résultera donc une force motrice, qui pousse la couche Y Z zy en arrière, & qui sera $= \frac{1}{4}\pi zz\,dp$: à laquelle ajoutant la force, qui résulte du poids de la couche qui étoit $= \frac{1}{4}\pi zz\,dy$, cette couche sera conjointement poussée en arrière par la force motrice $\frac{1}{4}\pi zz\,(dp + dy)$, qui étant divisée par la masse même de la couche $\frac{1}{4}\pi zz\,ds$, donne la force accélératrice $= \frac{dp + dy}{ds}$, qui étant contraire à la direction YY', il faut égaler V à $-\frac{dp - dy}{ds}$.

De là nous obtiendrons cette équation

$$dp + dy = -\frac{\alpha\alpha\,ds}{zz} \cdot \frac{dv}{dr} + \frac{4\alpha^4 S\,ds}{z^4} \cdot v$$

How did he formulate the equation of motion - five years before the „Euler equations"?

In this special case, the force which accelerates the fluid is exerted by a piston, driven by a constant weight from position MN to mn. From this piston motion follows the motion of the water in the pipe from YZ to Y'Z'. Euler expressed the acceleration at YZ in terms of the piston's acceleration and the tube's diameter from YZ to Y'Z'.

Then he balanced the forces acting on an infinitesimal slice of water at YZ.

Because Euler expressed the acceleration in terms of the piston's motion, the equation of motion looks strange.

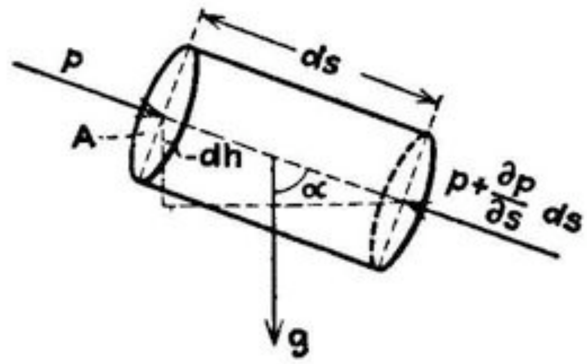


FIG. 2.—Forces on an element of ideal fluid.

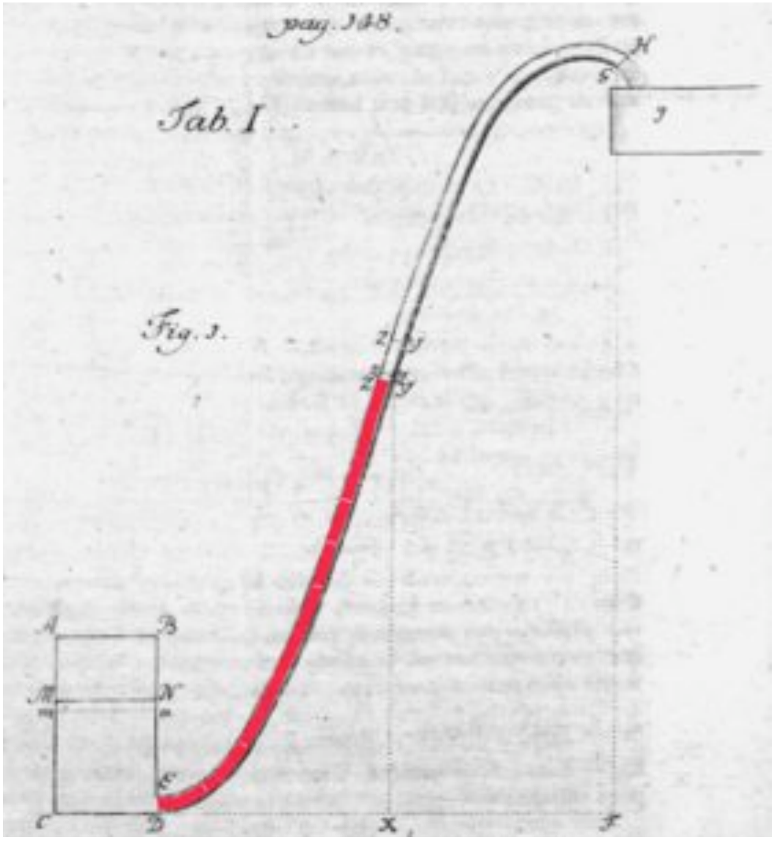
$$\underbrace{\rho dA ds \cdot \frac{Dw}{dt}}_{\text{mass} \times \text{acceleration}} = \underbrace{\rho g dA ds \cos \alpha}_{\text{gravity force}} + \underbrace{dA\left\{p - \left(p + \frac{\partial p}{\partial s} ds\right)\right\}}_{\text{pressure force}}$$

$$\frac{\partial p}{\partial s} + g\rho \frac{\partial y}{\partial s} = -\rho\left(\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial s}\right)$$

In modern notation, Euler‘s balance of forces corresponds to this textbook derivation. The equation of motion which Euler derived corresponds to the one-dimensional Euler equation along a streamline.

Such a reconstruction, of course, is a bit ahistorical. But it facilitates to see the physical argument involved in Euler‘s pipeflow theory. It is basically the same argument which led him later to the establishment of the general equation of motion.

Integration



Integration of

$$\frac{\partial p}{\partial s} + g\rho\,\frac{\partial y}{\partial s} = -\rho\left(\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial s}\right)$$

yields

$$p = p_0 + g\rho\,(y_0 - y) + \frac{\rho}{2}(w_0^2 - w_y^2) - \rho\int_0^Y \frac{\partial w}{\partial t}\,ds$$

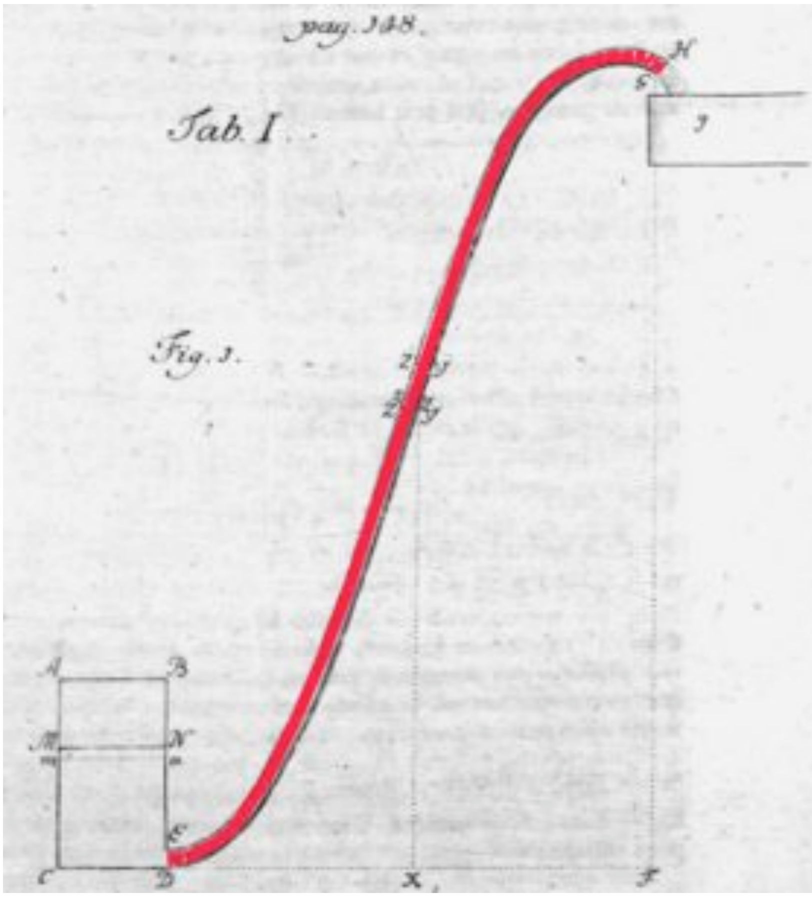
(nonstationary „Bernoulli Equation“)

Determine integration constants for the interior of the pump at *CD*:

$$p = (k + b - r - y)g\rho + \frac{\rho}{2}(w_p^2 - w^2) - \rho\,\frac{dw_p}{dt}(b - r) - \rho\int_D^Y \frac{\partial w}{\partial t}\,ds$$

Next Euler integrated the equation of motion and arrived at the Bernoulli equation for nonstationary flow. He thus was able to express the pressure at an arbitrary site of the pipeline in terms of the other quantities which entered the problem.

$$p = (k + b - r - y)g\rho + \frac{\rho}{2}(w_p^2 - w^2) - \rho\frac{dw_p}{dt}(b - r) - \rho\int_D^Y \frac{\partial w}{\partial t} ds$$



$p = 0$ at GH yields a differential equation for the velocity of the piston; its solution is:

$$r = \frac{L}{\frac{a^4}{c^4} - 1} \ln \frac{k - h}{k - h - \frac{w_p^2}{2g}(\frac{a^4}{c^4} - 1)}$$

In the limit of $k - H >> \frac{w_p^2}{2g}(\frac{a^4}{c^4} - 1)$:

$$w_p = \sqrt{\frac{2g(k - h)}{L} r}\ .$$

$$p(y) = g\rho\{k - y + b - r + \frac{k - h}{L}(1 - \frac{a^4}{z^4} - b + r) - \frac{(k - H)}{L} a^2 \int \frac{ds}{z^2}\}$$

$$\approx g\rho\{k - y - \frac{(k - H)}{L} a^2 \int \frac{ds}{z^2}\}$$

The formula for the pressure contained the velocities of the piston and in the tube, which of course are related to another by the continuity equation. Furthermore, by specifying the pressure at the upper end of the pipeline, a differential equation could be derived for the velocity of the piston. Euler solved these secondary problems (which involved some tedious mathematics) and finally was able to arrive at a formula for the pressure in terms of known quantities.

Further problems

PROBLEME III.

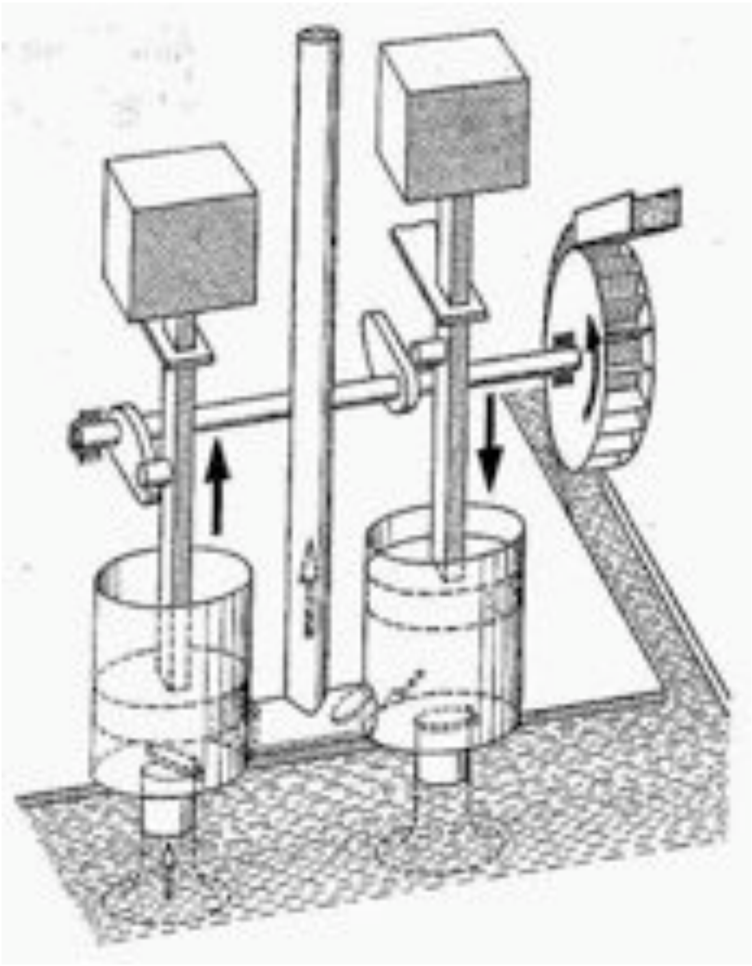
XIX. *La hauteur du réservoir* FG = *g étant extrêmement grande par rapport à la hauteur de la pompe, déterminer le tems, pendant lequel le piston est poussé en bas depuis* AB *presqu'en* E, *lorsqu'il est poussé par la force constante* = K.

PROBLEME IV.

XX. *Si deux pompes égales, dont les pistons sont aussi poussés par des forces égales, agissent alternativement, de sorte que pendant qu'une aspire l'autre refoule, & que toutes les deux refoulent l'eau dans le même tuyau montant* DEGH, *pour la dégorger dans le reservoir* I *élevé à une hauteur fort grande, déterminer le tems du jeu des pistons, la pression que le tuyau a à soutenir, & la quantité d'eau, qui sera fournie dans le réservoir pendant une heure.*

PROBLEME V.

XXXIV. *Le tems du jeu des pistons étant donné, avec les mesures de toutes les parties de la machine destinée à élever de l'eau, trouver la force qui doit agir sur les pistons, & la pression, que le tuyau montant aura à soutenir en bas.*



He did not content himself with this result, but applied it to solve further problems which addressed further practical issues, such as the use of a pump with two pistons.

qui ont entrepris la construction d'une telle machine ; puisque les tuyaux ne manqueront pas de crêver, quoiqu'on ait crû avoir pris toutes les précautions pour prévenir cet accident facheux. Je rapporterai un exemple, d'où l'on verra combien la pression sur le tuyau peut devenir grande au delà de la hauteur simple de l'eau dans le tuyau.

EXEMPLE.

XLIII. La machine proposée avoit ces mesures.

Le diametre des pompes	=	$\frac{4}{3}$ pieds	=	a
La hauteur du jeu des pistons	=	4 pieds	=	b
Le diametre du tuyau montant	=	$\frac{3}{4}$ pieds	=	c
La longueur du tuyau	=	3000 pieds	=	l
La hauteur du tuyau	=	60 pieds	=	g.

Chaque jeu des pistons s'achevoit en 6 secondes. Cela posé, on demande la pression, que le tuyau dût soutenir en bas.

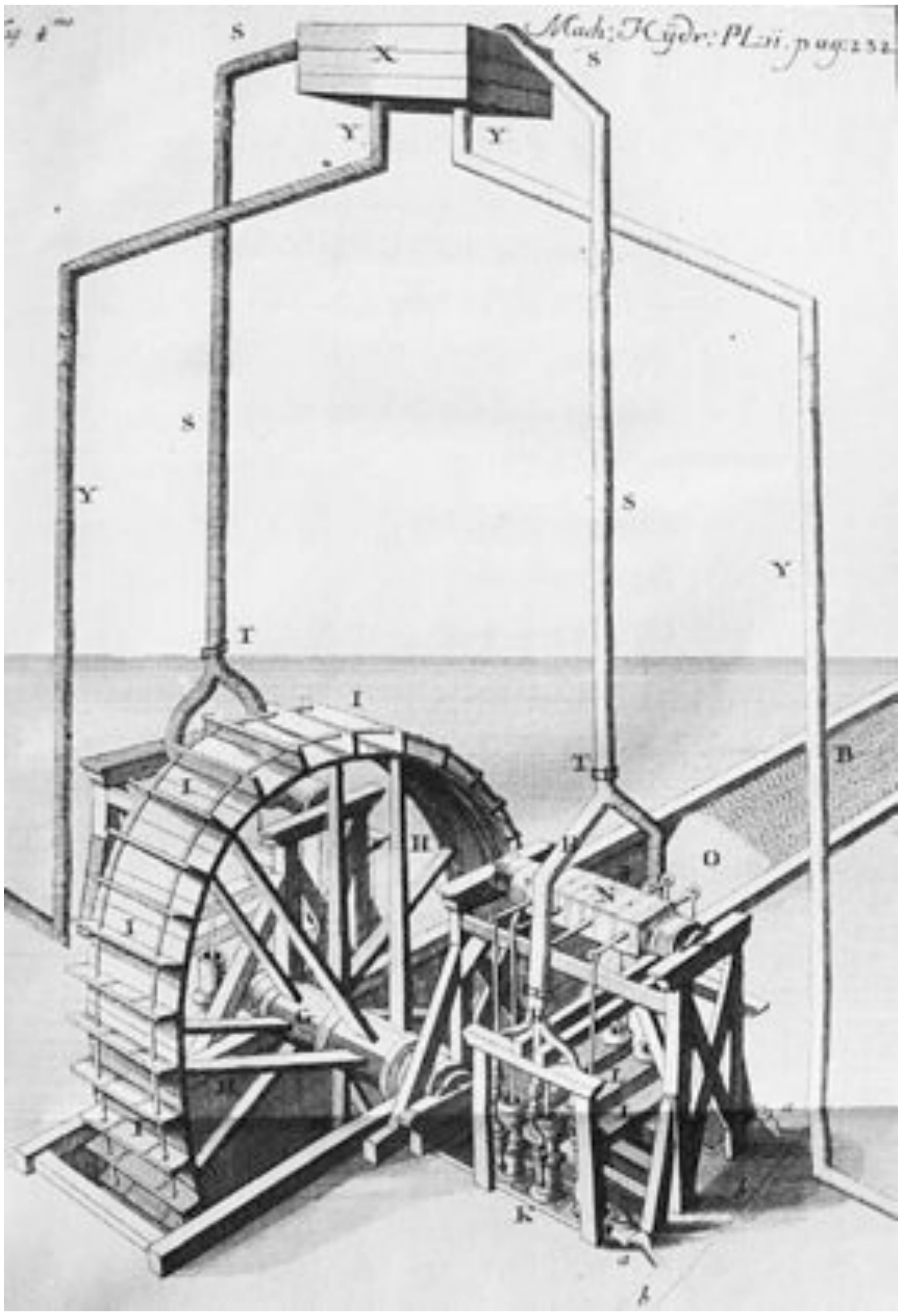
Ayant donc $t = 6''$ & posant cette pression équivalente à la hauteur p, on aura :

$$p = 60 + \frac{0,256.\ \frac{4^4}{3^4}.\ 4.\ 3000}{\frac{3^2}{4^2}.\ 36}\ \text{pieds.}$$

qui se réduit à $p = 60 + 270 = 330$ pieds.

Donc, si le tuyau n'avoit pas été assez fort pour porter une colomne d'eau de 330 pieds de hauteur, il feroit crévé infailliblement ; quoique la hauteur de l'élevation de l'eau ne fût que de 60 pieds, de sorte que le tuyau dût soutenir une force plus de 5 fois plus grande, que le simple poids de la colomne d'eau. De là on connoîtra la force, qui agit sur chaque piston, qui étoit $= \frac{\pi}{4} \cdot \frac{16}{9} \cdot 330 = 461$ pieds cubiques d'eau, & la quantité d'eau élevée dans une heure = 6701 pieds cubiques.

DIS



He concluded his memoir with a numerical example: For a discharge of 6701 cubic feet per hour, pumped to a height of 60 feet through a 3000 feet long pipeline, the pressure at the lower end of the pipeline amounted to an equivalent height of a 330 feet high water column. If the pipeline would have been supposed to withstand only the hydrostatic pressure, Euler warned, it would have inevitably burst. I

In his letter to the King on 17 October 1749, Euler had presented the same lesson – here with direct reference to the mishap at Sanssouci (which he did not mention in his memoir) when the pipes burst at the first trials.

Euler also formulated the practical lessons from this result verbally as rules:

„Pour que la meme force qui agit sur les pistons des pompes soit en état de fournir dans le réservoir la plus grande quantité d'eau, il faut avoir soin de faire le tuyau montant aussi large qu'il sera possible (...) Pour fournir une plus grande quantité d'eau dans le réservoir par la meme force qui agit sur les pistons, il faut rendre le tuyau montant aussi court qu'il sera possible."

Without knowing why, these rules were implicitely followed by many contemporary water art projects, such as in Bavaria at Nymphenburg (right image). But the practitioners at Sanssouci ignored such examples either.

Conclusion

- Euler‘s approach was problem-oriented; the Sanssouci-problem led him to formulate the equation of motion for this specific case.
- He integrated the equation (the one-dimensional „Euler equation“) and thus (re)derived the non-stationary „Bernoulli equation“.
- He derived from his solution practical rules which could have avoided the mishaps at Sanssouci: But his advice was ignored.
- When he proceeded to the general case, he relied on these experiences with practical problems: Ideal flow theory emerged from practical real flow problems