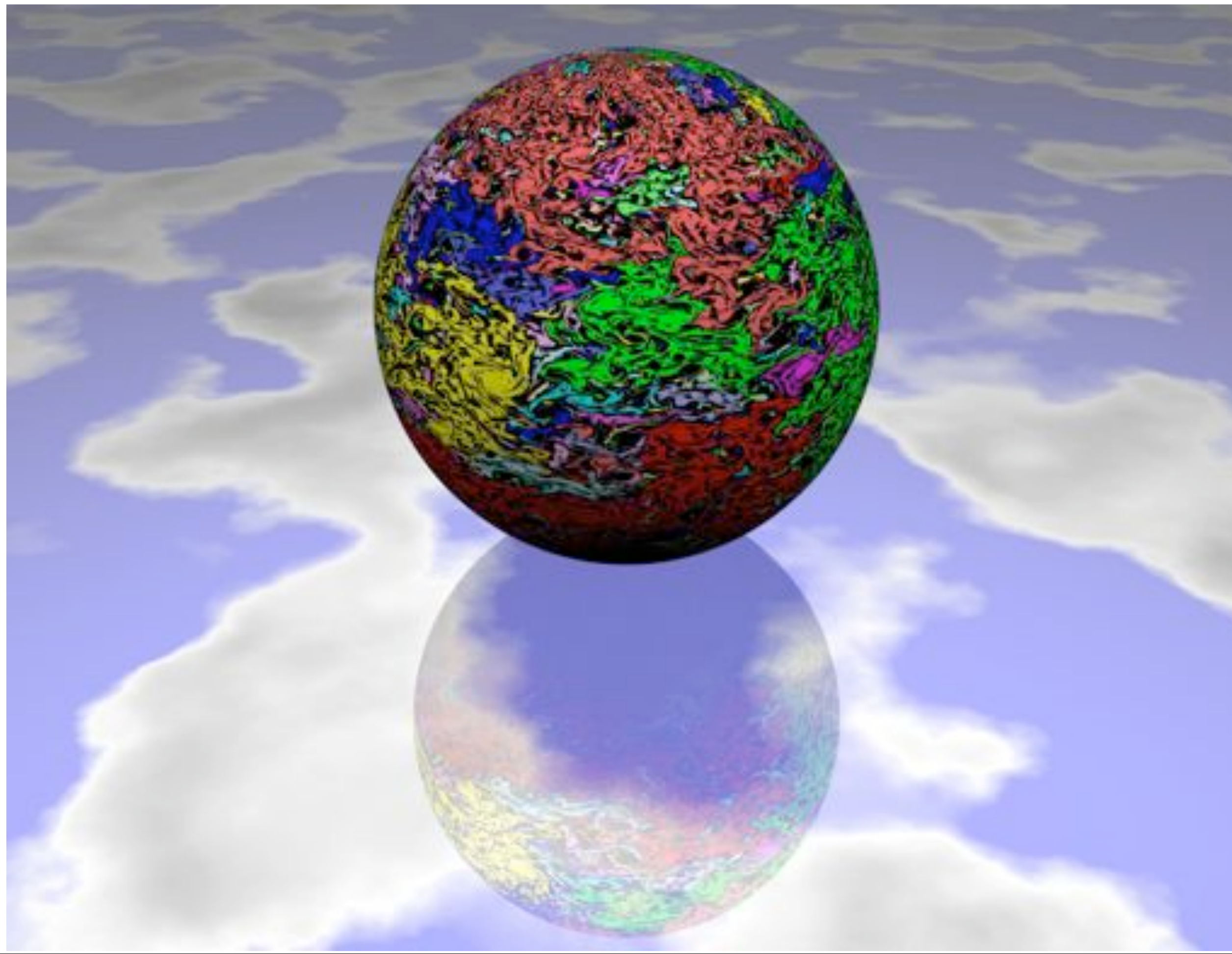
Conformal invariance and 2d turbulence

G. Falkovich



Euler equations: 250 years on

Aussois, June 2007

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Conformal invariance in two-dimensional turbulence

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Inverse Turbulent Cascades and Conformally Invariant Curves

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Euler equation in 2d describes transport of vorticity

$$\omega = \nabla \times \mathbf{v}$$

$$\partial_t \omega + (\mathbf{v} \cdot \nabla)\omega = 0$$

$$\mathbf{v} = (\partial\Psi/\partial y, -\partial\Psi/\partial x)$$

$$\Psi(\mathbf{x}, t) = \int d\mathbf{y} \, \log|\mathbf{x} - \mathbf{y}| a(\mathbf{y}, t)$$

Family of transport-type equations

$$\partial_t a + (\mathbf{v} \cdot \nabla) a = 0$$

$$\mathbf{v} = (\partial\Psi/\partial y, -\partial\Psi/\partial x)$$

$$\Psi(\boldsymbol{r}, t) = \int d\boldsymbol{r}' \, |\boldsymbol{r} - \boldsymbol{r}'|^{m-2} a(\boldsymbol{r}', t)$$

This system describes geodesics on an infinitely-dimensional Riemannian manifold of the area-preserving diffeomorfisms. On a torus,

$$a_{\mathbf{k}}(t) = \int a(\mathbf{x}, t) e^{\imath(\mathbf{k}\cdot\mathbf{x})} d\mathbf{x}$$

$$\partial a_{\mathbf{k}}/\partial t = \sum_{\mathbf{j}} j^{-m} [\mathbf{k}, \mathbf{j}] a_{\mathbf{j}} a_{\mathbf{k}-\mathbf{j}} = \sum_{\mathbf{ijl}} \alpha^{\mathbf{ij}} C^{\mathbf{l}}_{\mathbf{kj}} a_{\mathbf{i}} a_{\mathbf{l}}$$

$$[\mathbf{k}, \mathbf{j}] = k_1 j_2 - k_2 j_1$$

$$\text{inertia tensor, } \alpha^{\mathbf{ij}} = j^{-m} \delta_{\mathbf{i}+\mathbf{j}}$$

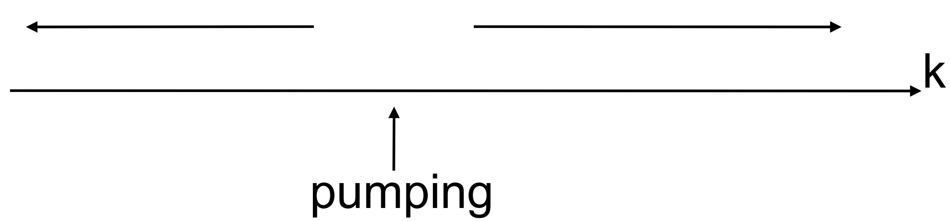
Add force and dissipation to provide for turbulence

$$\partial a/\partial t + (\mathbf{v}\cdot\nabla)a = f + \nu\Delta a - \alpha a \qquad (*)$$

$$\frac{\partial a_{\mathbf{k}}}{\partial t} - \sum_{\mathbf{j}} [\mathbf{k},\mathbf{j}] j^{-m} a_{\mathbf{j}} a_{\mathbf{k}-\mathbf{j}} = f_{\mathbf{k}} - (\alpha + \nu k^2) a_{\mathbf{k}}$$

lhs of (*) conserves

$$\sum |a_{\mathbf{k}}|^2 k^{-m} \qquad\qquad \sum |a_{\mathbf{k}}|^2$$



$$\langle f_{\mathbf{k}}(0) f_{\mathbf{k}'}(t)\rangle \;=\; D(\mathbf{k})\delta_{\mathbf{k}\mathbf{k}'}\delta(t)$$

$$k_f < k < Ak_f$$

$$\sum_{\mathbf{k}}(\alpha + \nu k^2)E\left[|a_{\mathbf{k}}|^2\right] = \sum D(\mathbf{k}) \equiv P\,,$$

$$\sum_{\mathbf{k}}(\alpha + \nu k^2)k^{-m}E\left[|a_{\mathbf{k}}|^2\right] = \sum D(\mathbf{k})k^{-m} \equiv Q$$

$$\nu \to 0 \qquad k_\nu \gg k_f$$

$$k_f \to \infty \qquad k_\alpha \ll k_f \qquad k_\alpha \simeq (\alpha^3/Q)^{1/(4-m)}$$

$$k_\alpha \ll k \ll k_f \qquad E\left[|a_{\mathbf{k}}|^2\right] = Q^{2/3}k^{(4m-10)/3}$$

$$E\left[a_r^2\right] = \int \langle |a_{\mathbf{k}}|^2\rangle (1 - e^{\imath(\mathbf{k}\cdot\mathbf{r})})\, d\mathbf{k} \propto Q^{2/3} r^{(4-4m)/3} \propto r^{2h}$$

m=2

2d Navier-Stokes equations

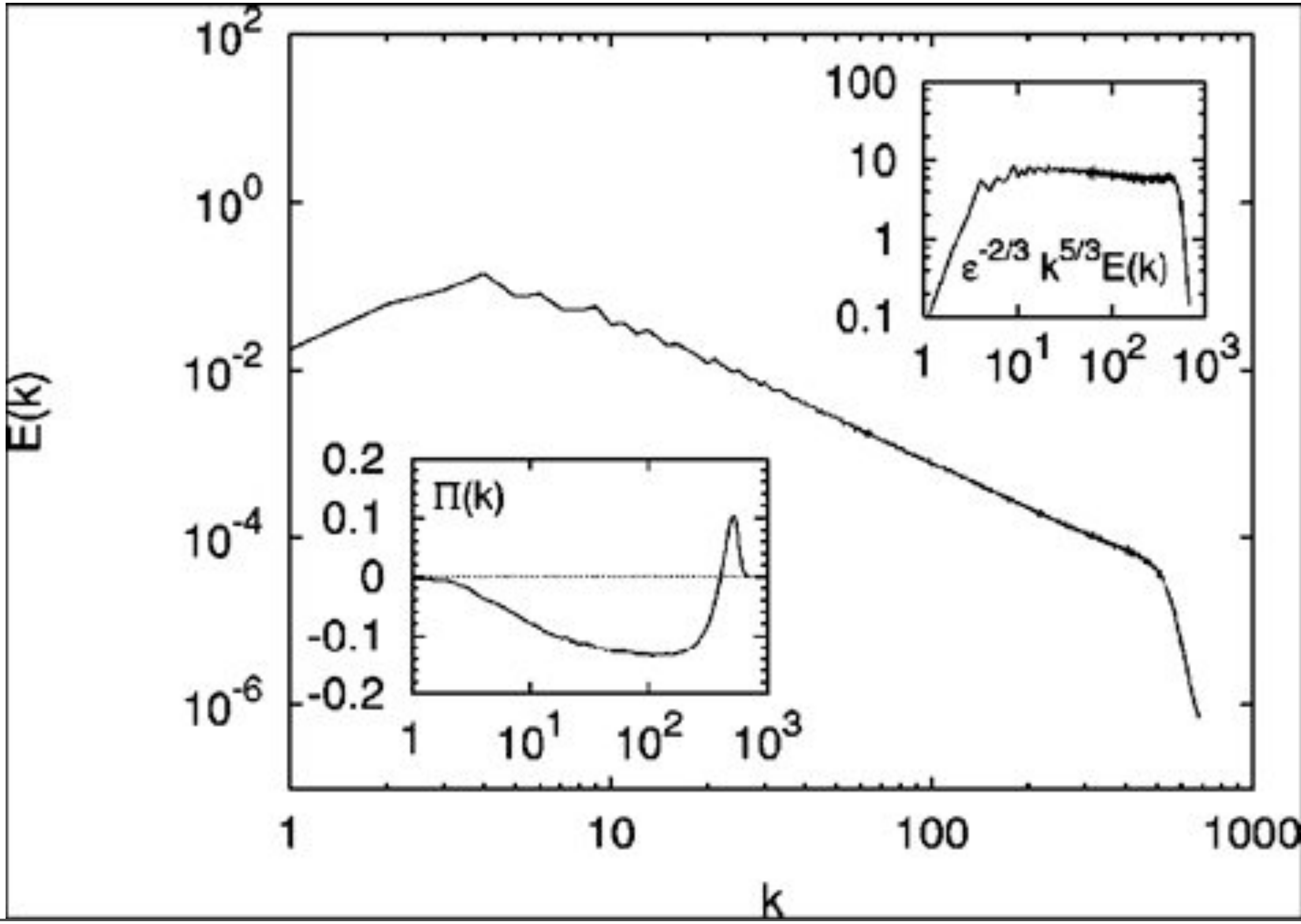
Kraichnan 1967

$$\omega = \nabla \times \mathbf{u}$$

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = \nu \Delta \omega - \alpha \omega + \nabla \times f$$

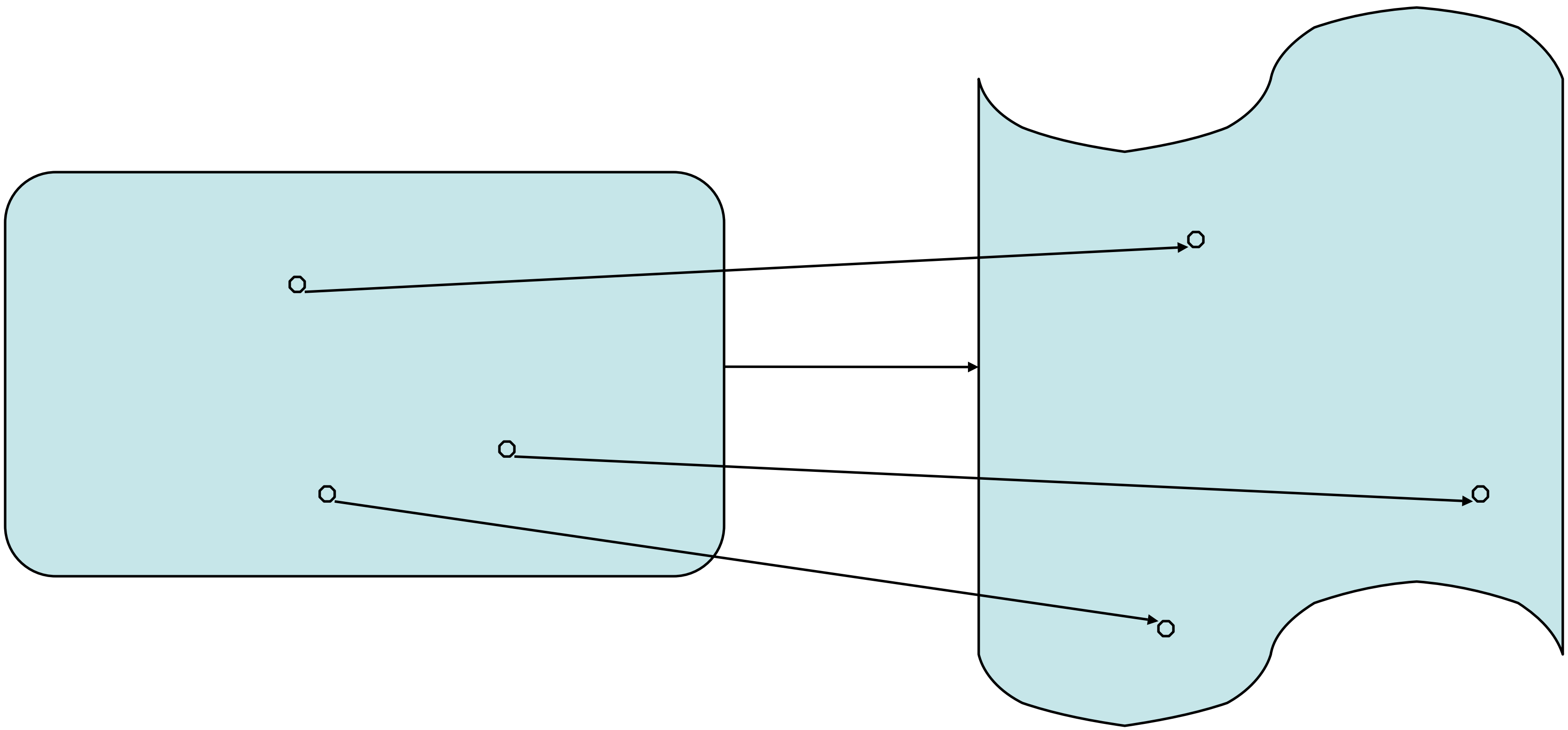
$$E = \frac{1}{2}\int |\mathbf{u}|^2 d^2x$$

$$Z = \frac{1}{2}\int \omega^2 d^2x$$

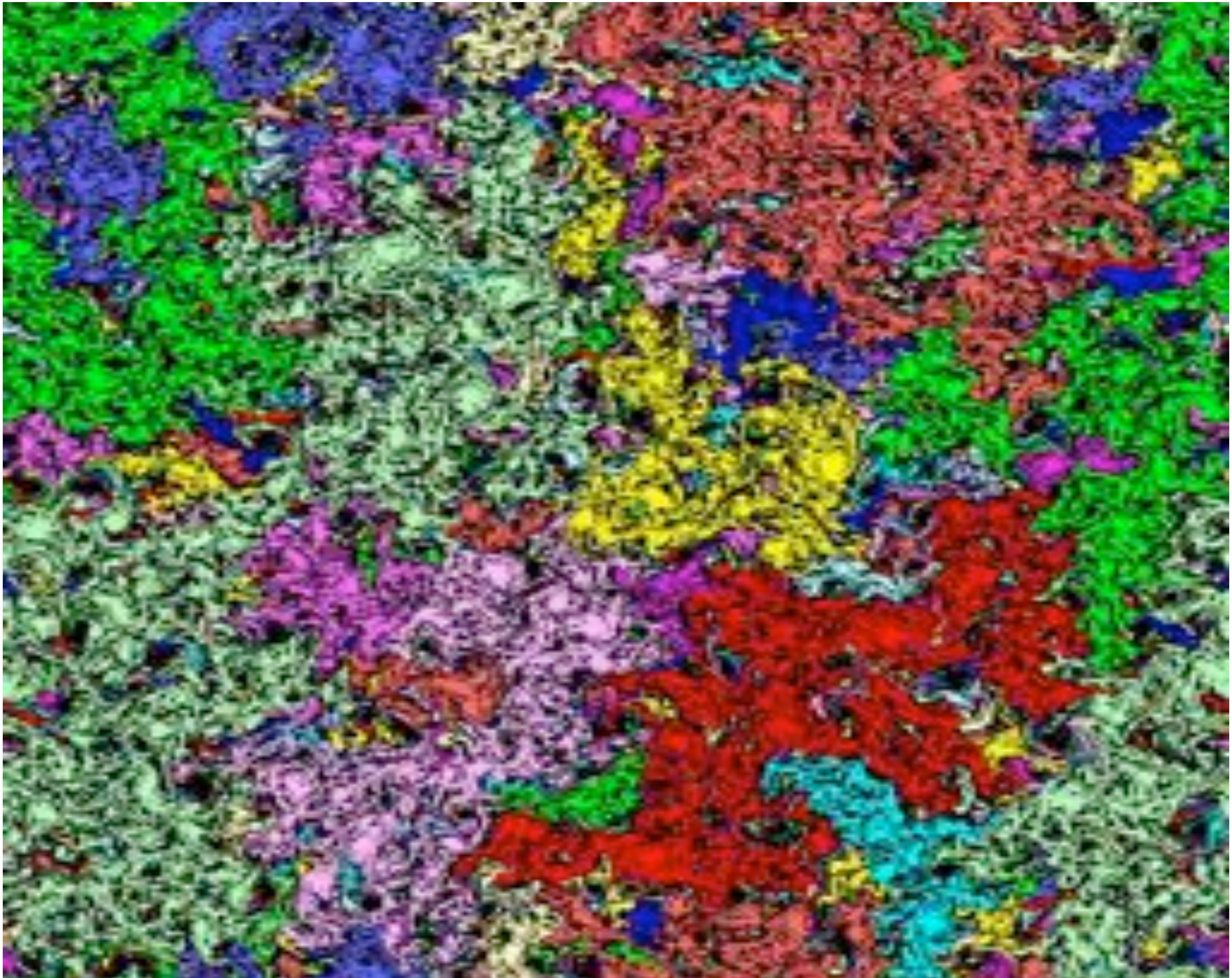


Strong fluctuations - infinitely many strongly interacting degrees of freedom → scale invariance.
Locality + scale invariance → conformal invariance

$$\mathcal{P}(\delta v, r) = (\delta v)^{-1} f(\delta v / r^h)$$



$$\mu_{\mathcal{D}}(z_1, \ldots, z_n) = \mu_{\mathcal{D}'}(f(z_1), \ldots, f(z_n))$$



Kolmogorov-Kraichnan scaling in 2d.

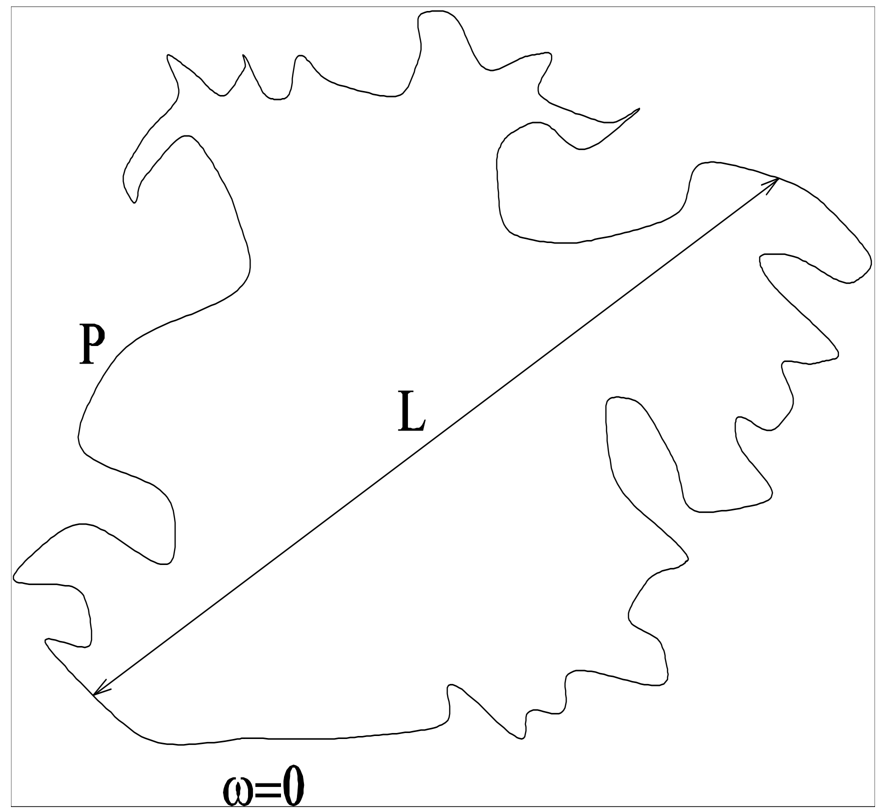
$$\frac{\text{kinetic energy } v_r^2}{\text{time } r/v_r} = \text{energy flux, } \epsilon$$

$$v_r^3 \sim \epsilon r$$

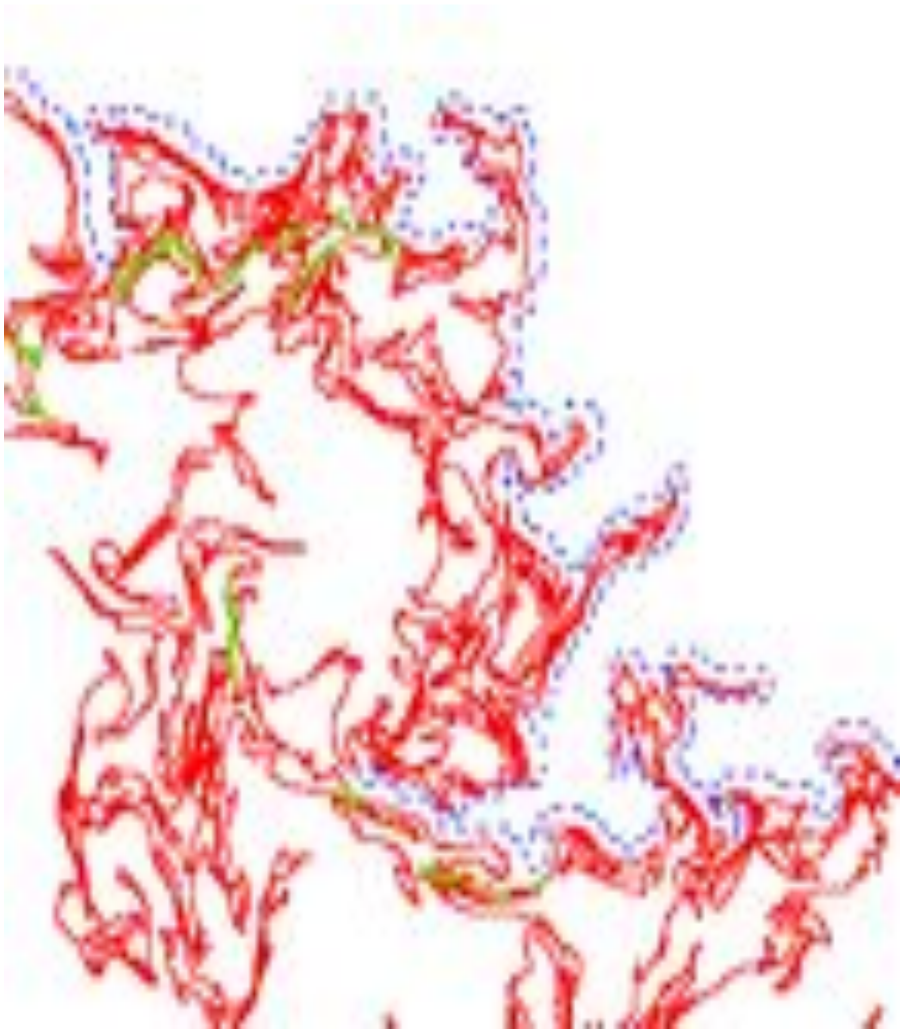
$$\int \omega dS \sim \omega_L L^2 \quad \propto L^{4/3}$$

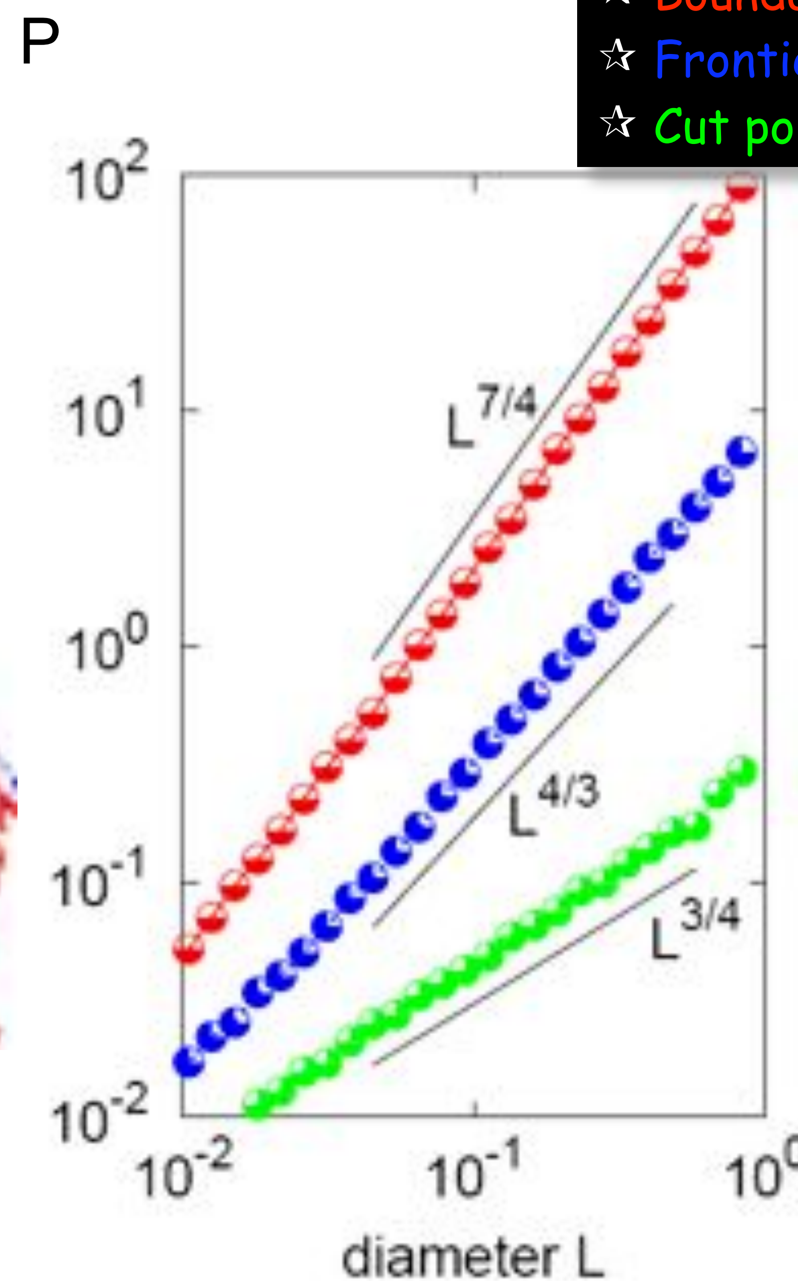
$$\Gamma = \oint \boldsymbol{v} \cdot d\boldsymbol{\ell} \propto P$$

$$P \propto L^{4/3}$$

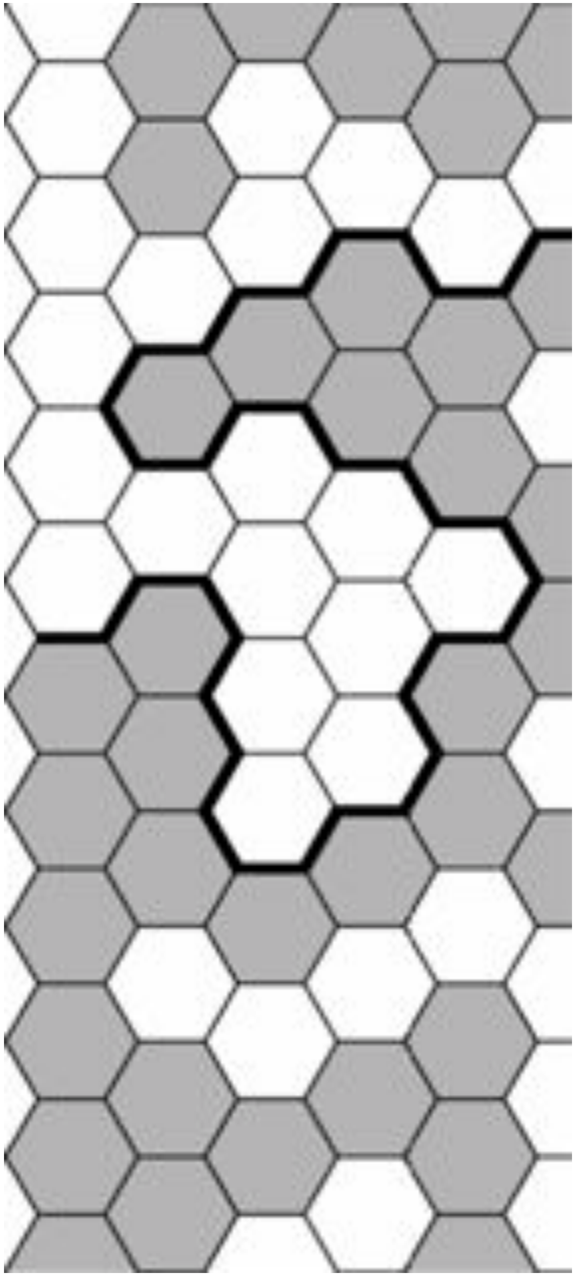


☆ Boundary
☆ Frontier
☆ Cut points

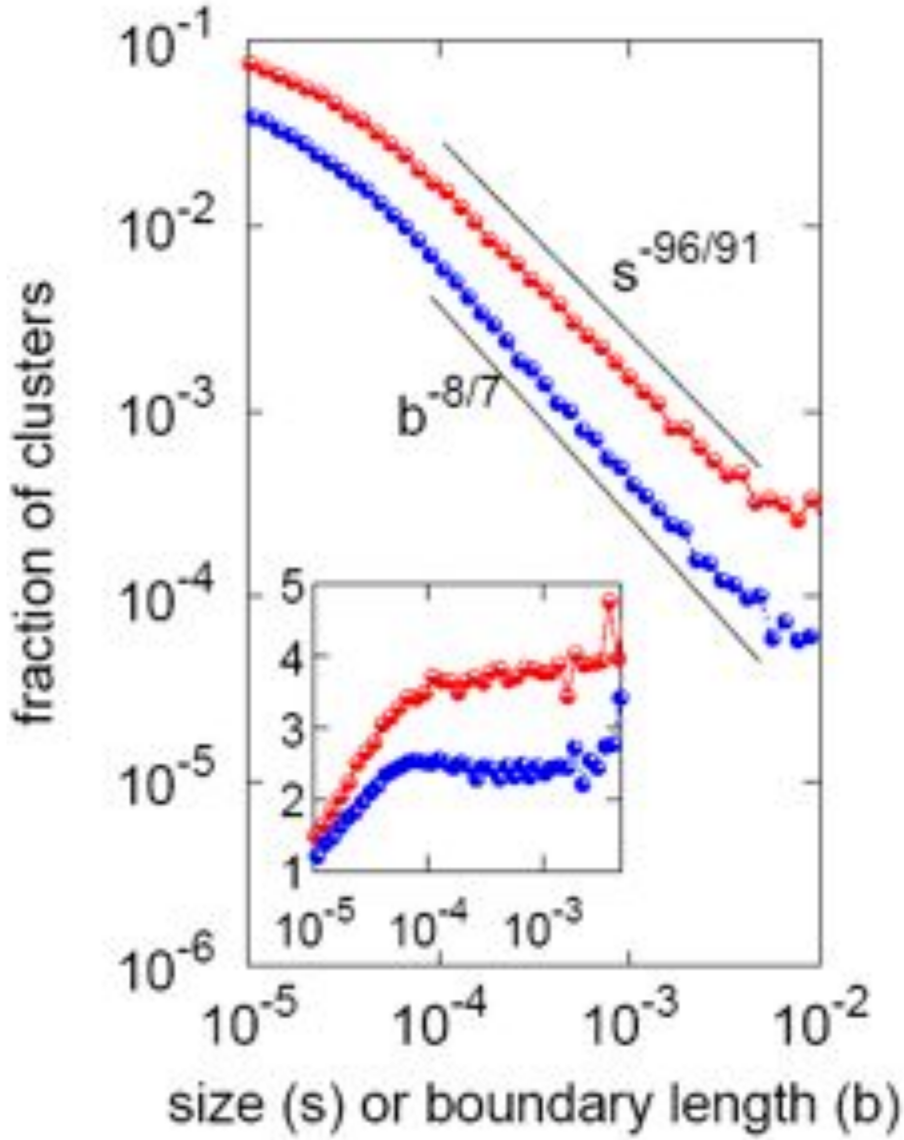




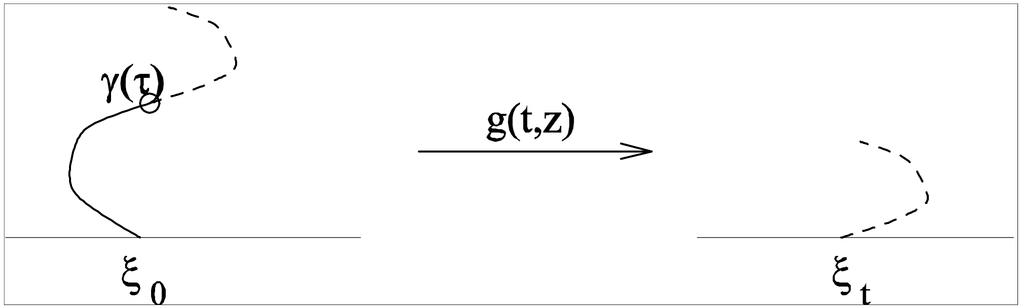
Critical Percolation



Vorticity clusters



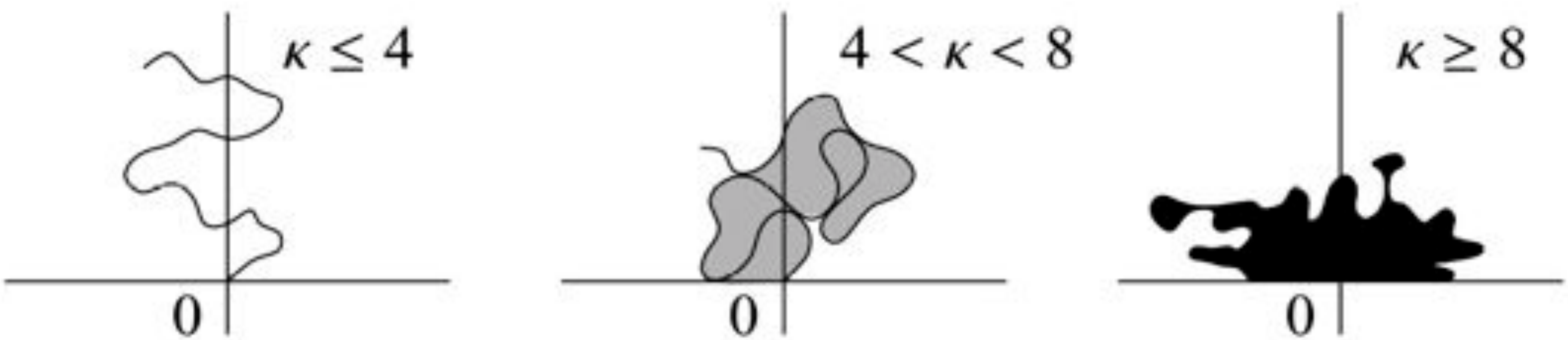
Schramm-Loewner Evolution (SLE)



$$g_t(z) \sim z + 2t/z + \mathrm{O}(1/z^2) \ \text{at infinity.}$$

$$dg_t(z)/dt = 2[g_t(z) - \xi(t)]^{-1}$$

$$\langle(\xi(t) - \xi(0))^2\rangle = \kappa t.$$



fractal dimension of SLE_κ

$$D_\kappa = 1 + \kappa/8$$

exterior perimeter of SLE_κ with $\kappa > 4$ is conjectured

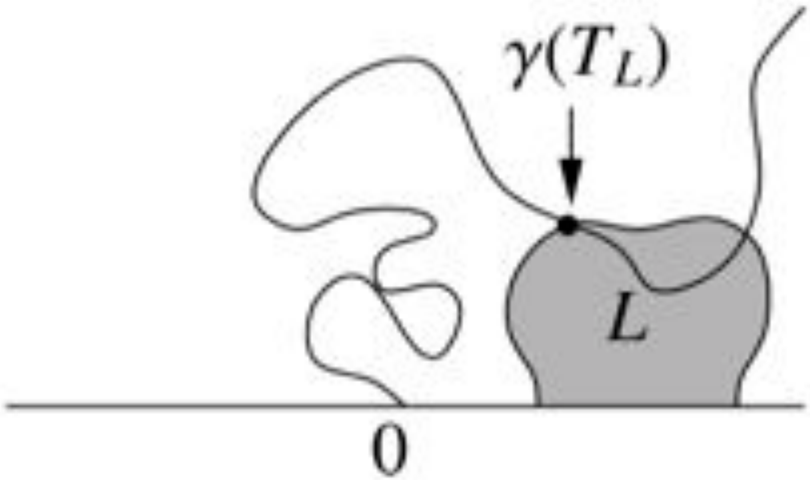
to look locally as SLE_{κ_*} curve with $\kappa_* = 16/\kappa$

$$(D-1)(D_*-1) = 1/4$$

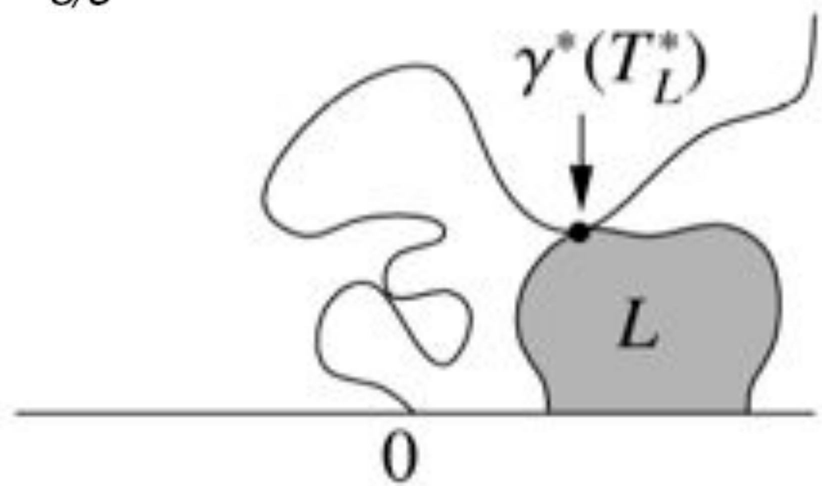
$$D = 7/4 \qquad\qquad D_* = 4/3$$

$\kappa = 6$ and $\kappa_* = 8/3$ correspond to CFT with zero central charge

Locality Property of SLE_6



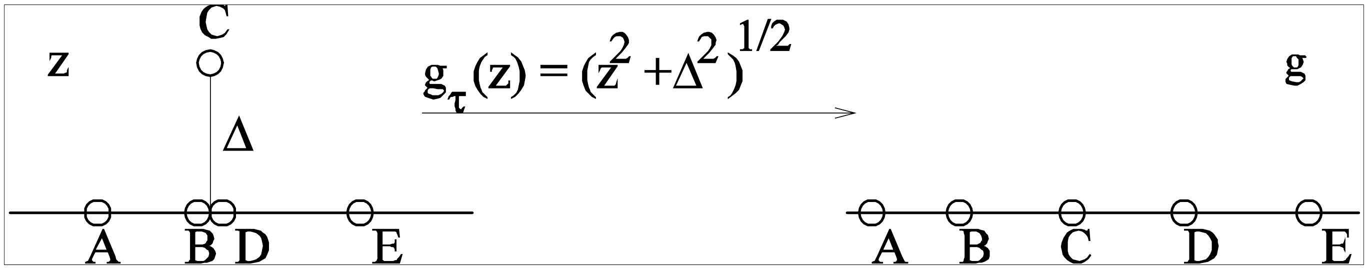
Restriction Property of $SLE_{8/3}$



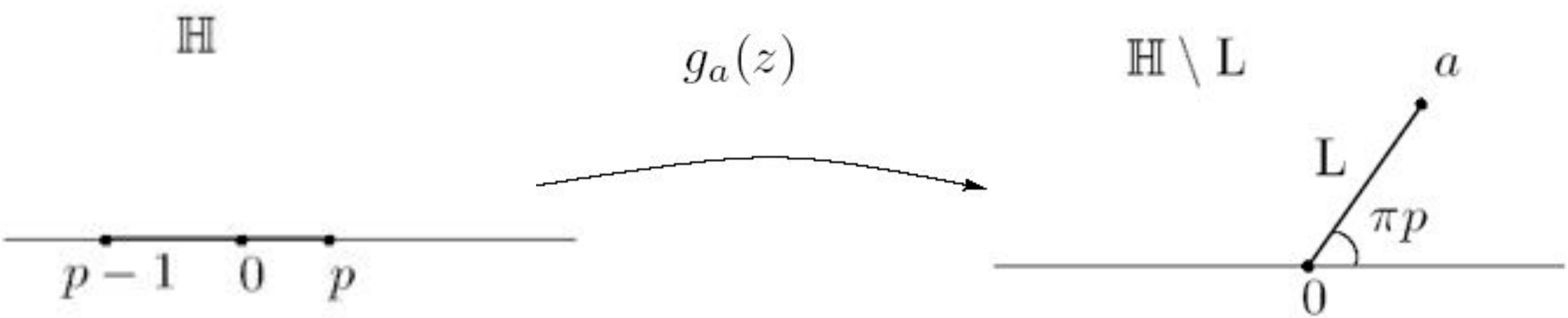
approximate $g_t(z)$ by a composition of

discrete, conformal slit maps

that swallow one segment of the curve at a time



C=ξ(t)



$$g_a(z) = C(z-p)^p(z+1-p)^{1-p}$$

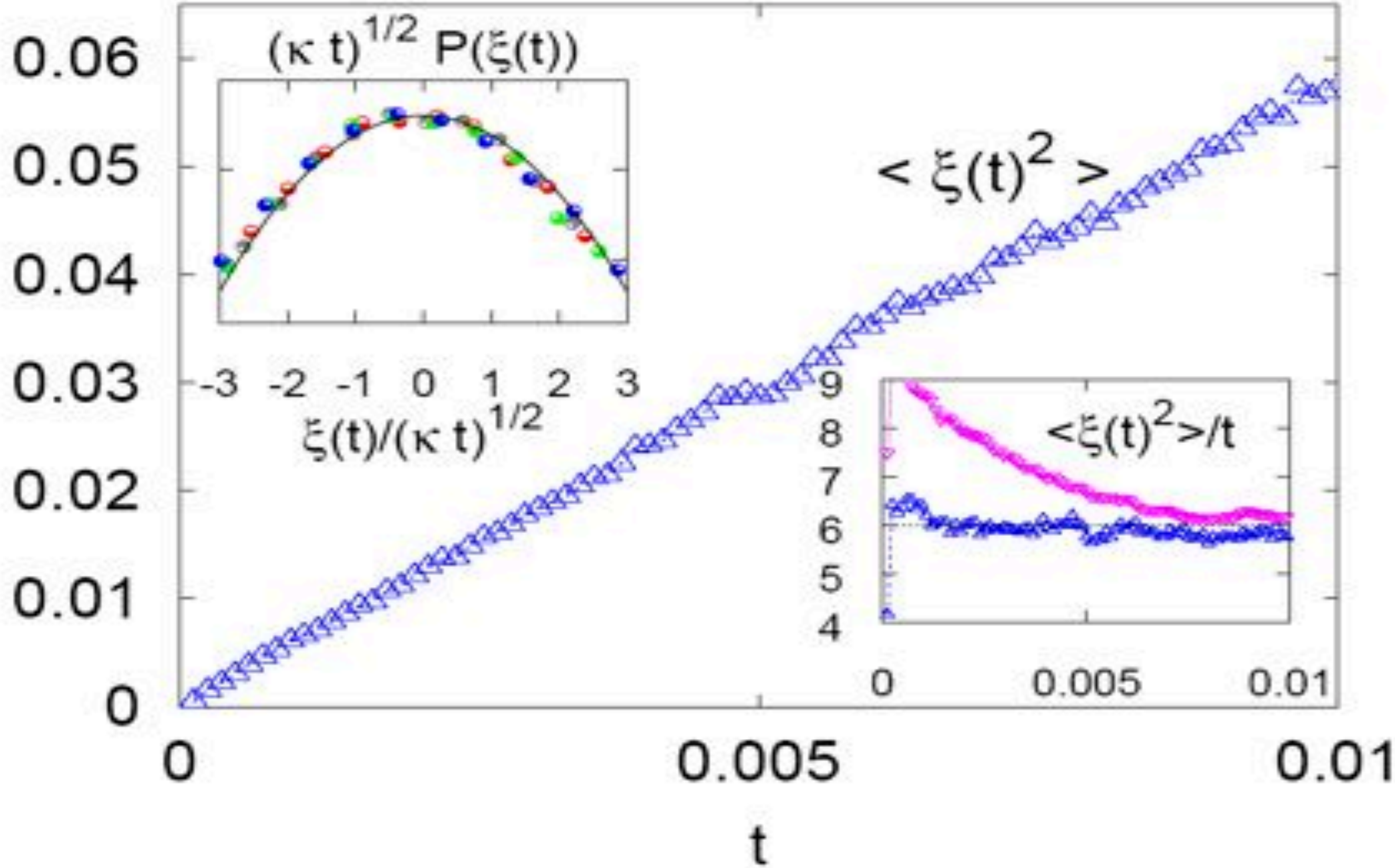
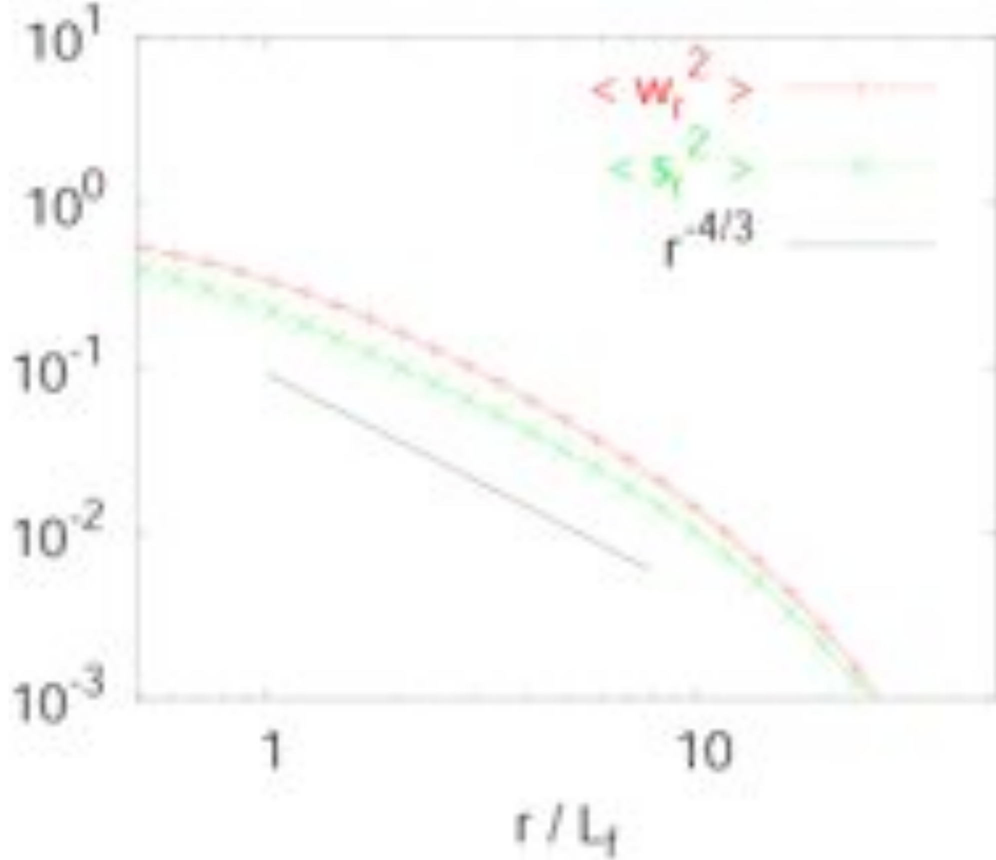
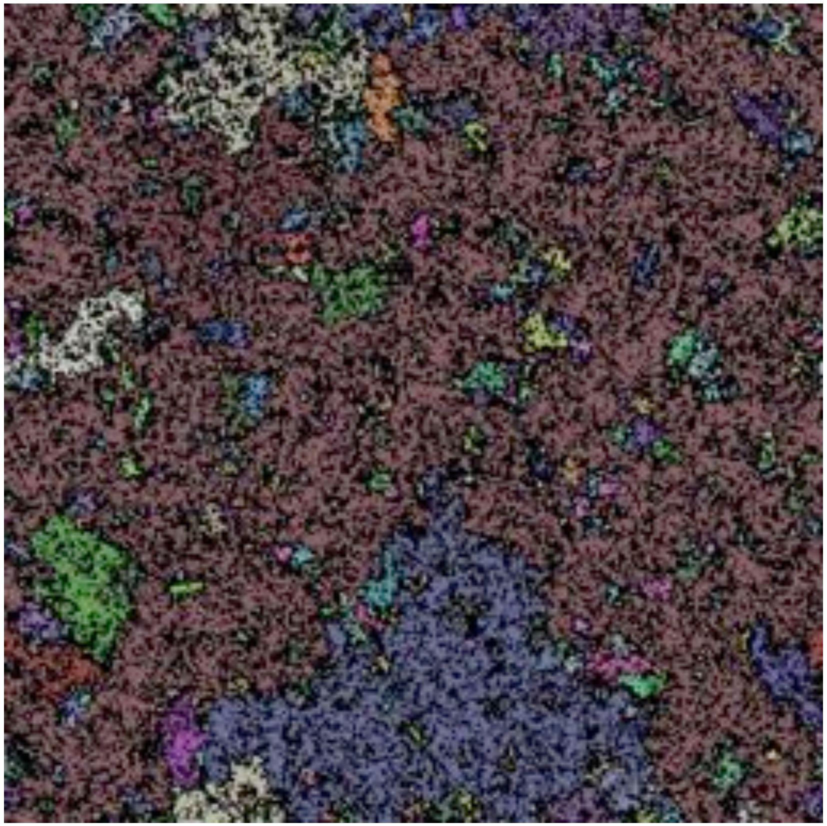


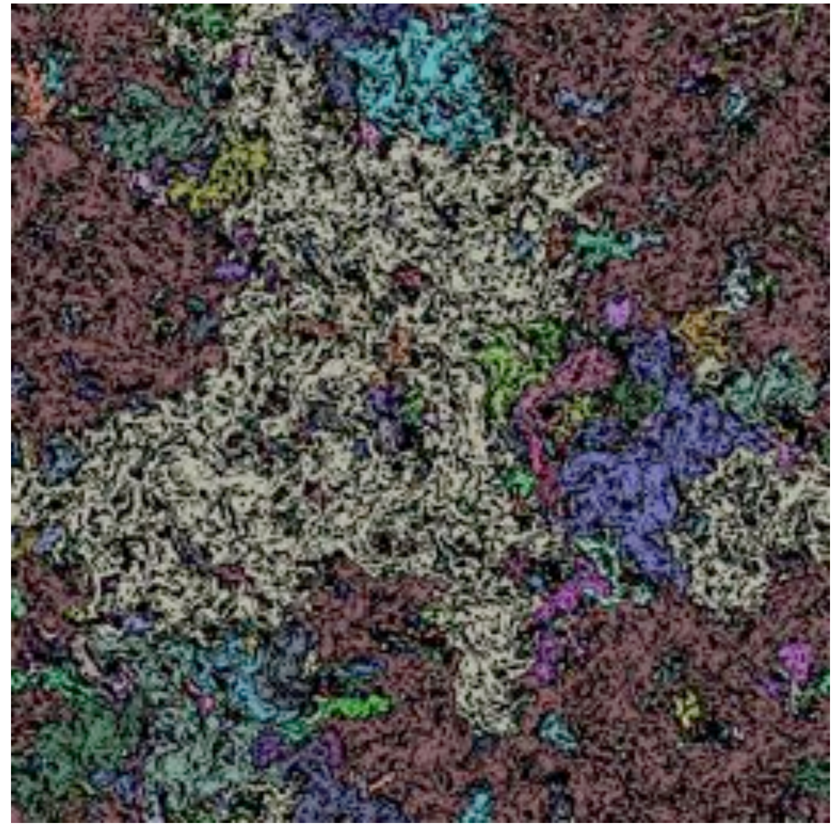
FIG. 4: The driving function is an effective diffusion process with diffusion coefficent $\kappa = 6 \pm 0.3$. The inverse cascade range corresponds to $5 \cdot 10^{-5} < t < 10^{-2}$. *Main frame*: the linear behaviour of $\langle \xi(t)^2 \rangle$. *Lower-right inset* : Diffusivity: blue for vorticity isolines, pink for the field with randomized phases. *Upper-left inset*: the probability density function of the rescaled driving function $\xi(t)/\sqrt{\kappa t}$ at four different times $t = 0.0012, 0.003, 0.006, 0.009$; the solid line is the Gaussian distribution $g(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.



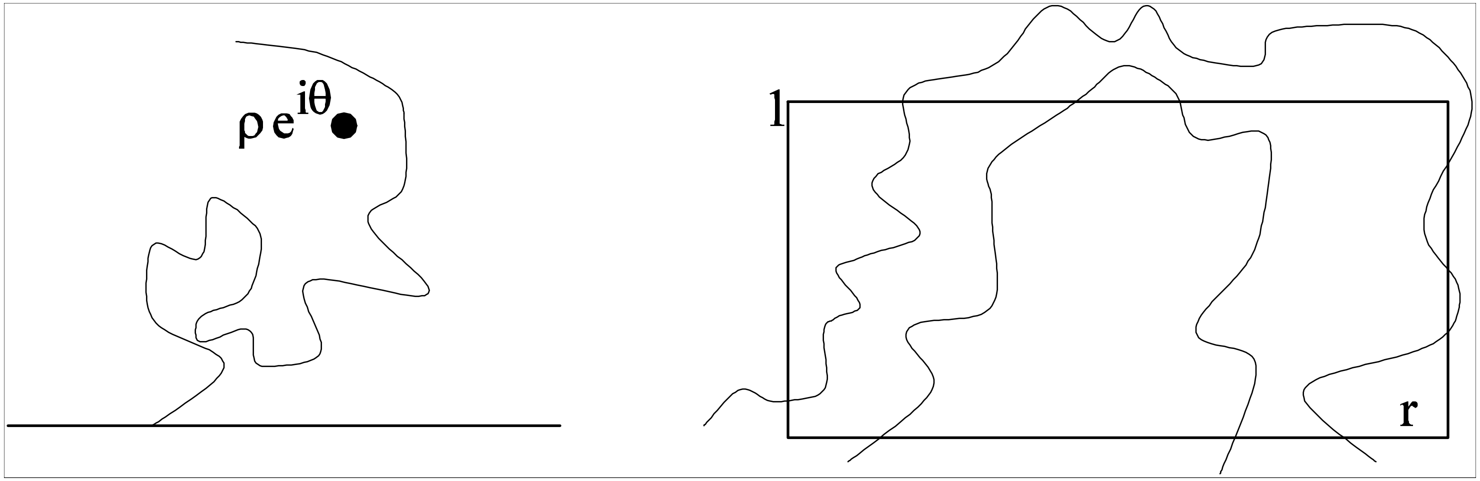
$\langle \omega(0)\omega(\mathbf{r}) \rangle \propto r^{-4/3}$ Harris criterion $r^{-3/2}$



Phase randomized



Original

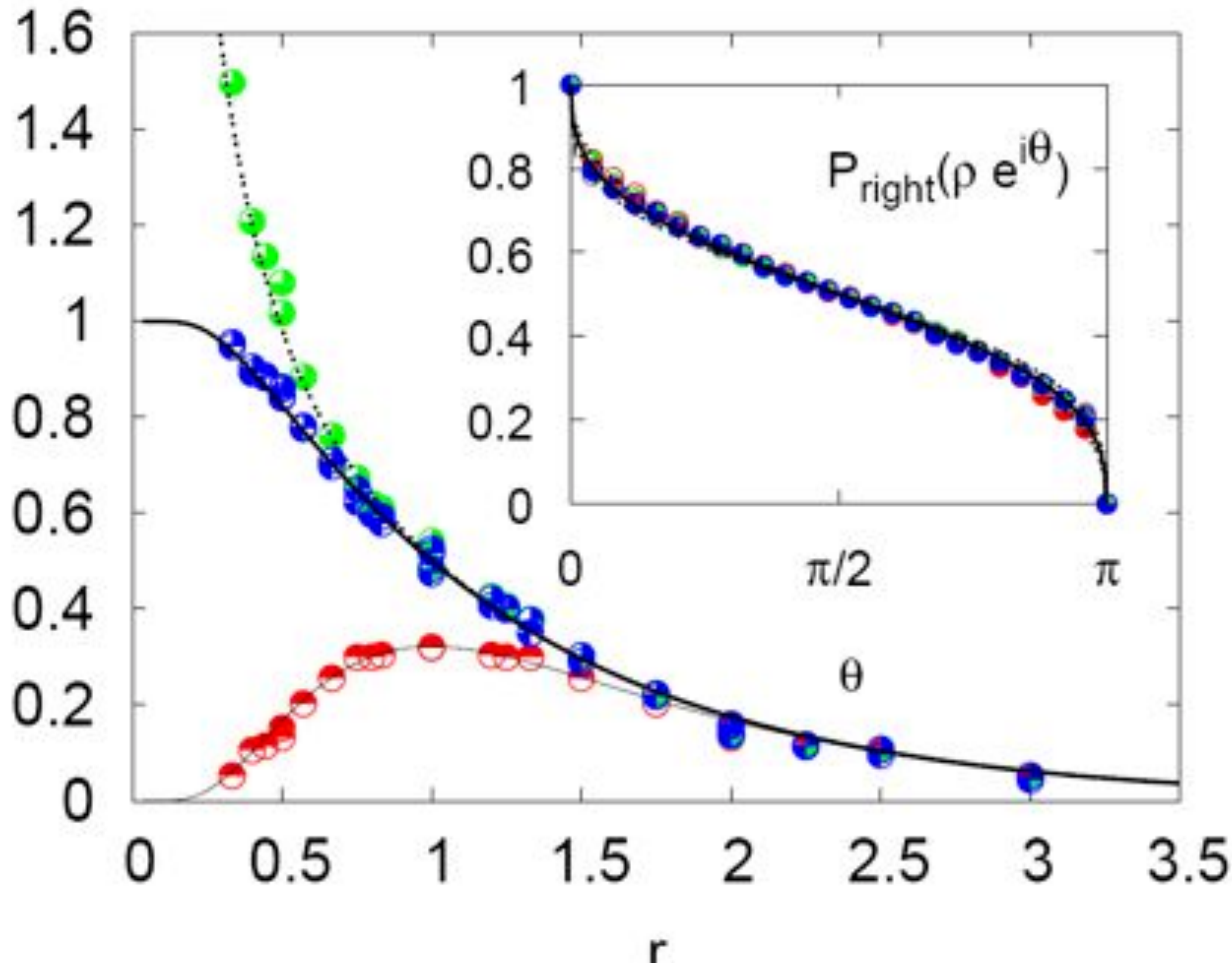


$$P = \frac{1}{2} + \frac{\Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\Gamma(\frac{8-\kappa}{2\kappa})}\ {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}; \frac{3}{2}; -\cot^2\theta\right)\cot\theta$$

$$\pi_v = \frac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2}\eta^{1/3}\ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \eta\right) \text{ with } \eta = [(1-k)/(1+k)]^2$$

$$r = K(1-k^2)/[2K(k^2)]$$

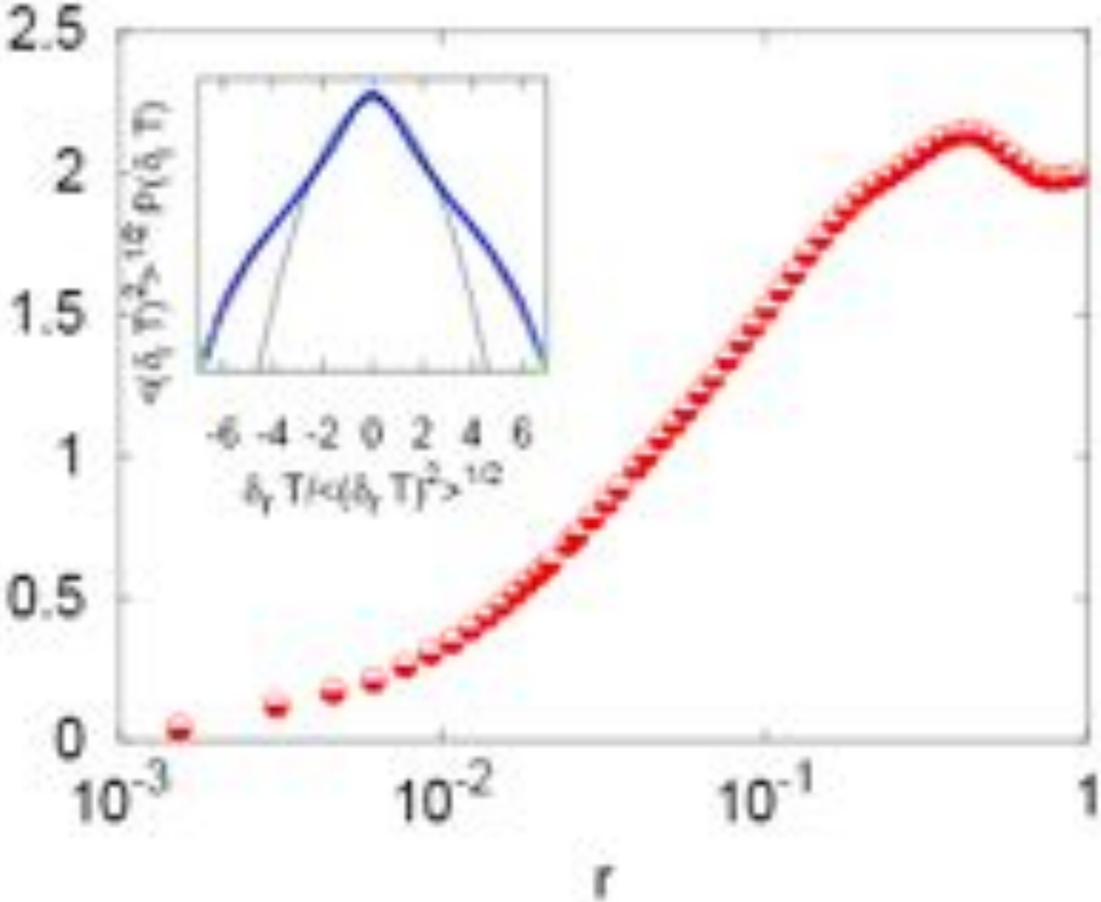
$$\pi_{hv} = \pi_v - \frac{\eta}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})}\ {}_3F_2\left(1, 1, \frac{4}{3}; 2, \frac{5}{3}; \eta\right)$$

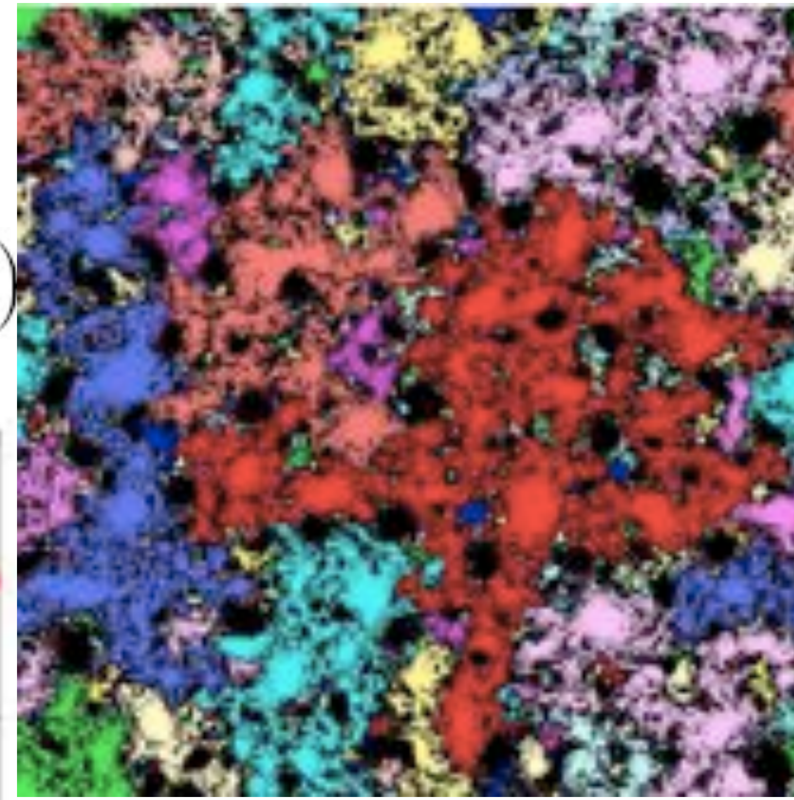


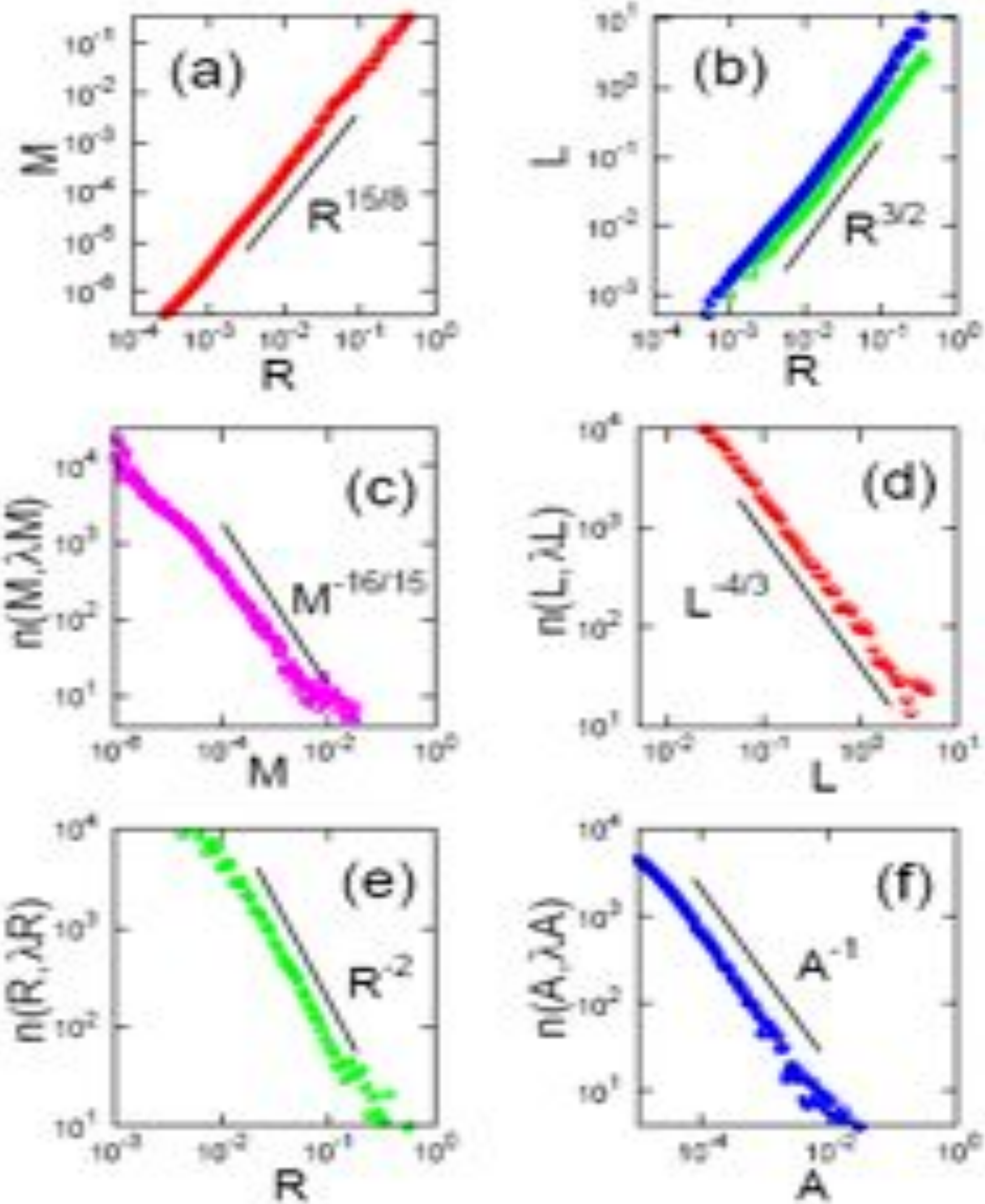
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
r
θ
$P_{right}(\rho\ e^{i\theta})$
1
0.8
0.6
0.4
0.2
0
0
π/2
π

$$m = 1,\ E\left[a_r^2\right] \propto \ln(k_f r)$$

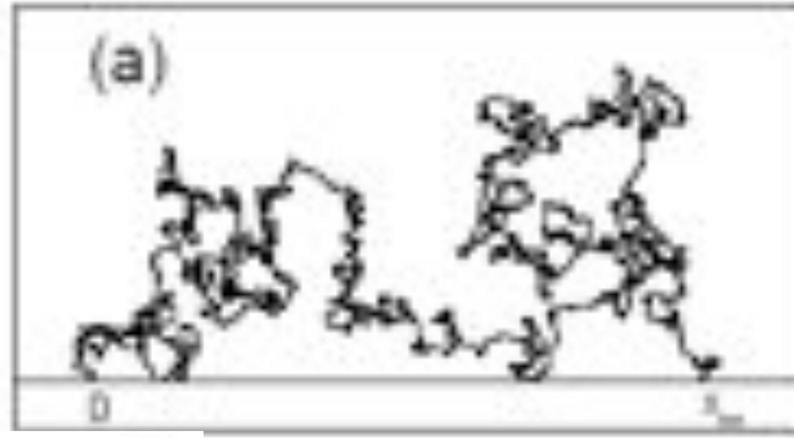
$$\mathcal{P}(a_r, r) \sim a_r^{-1} f(a_r / \ln(k_f r))$$



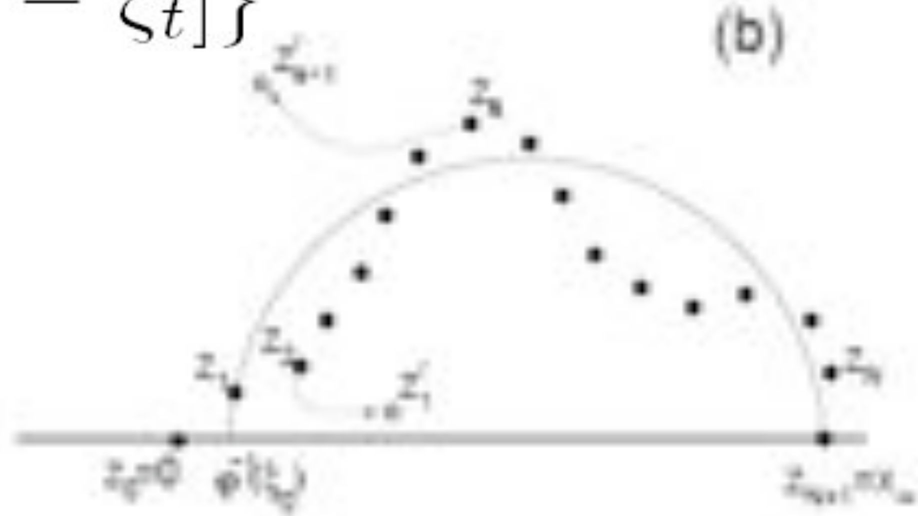




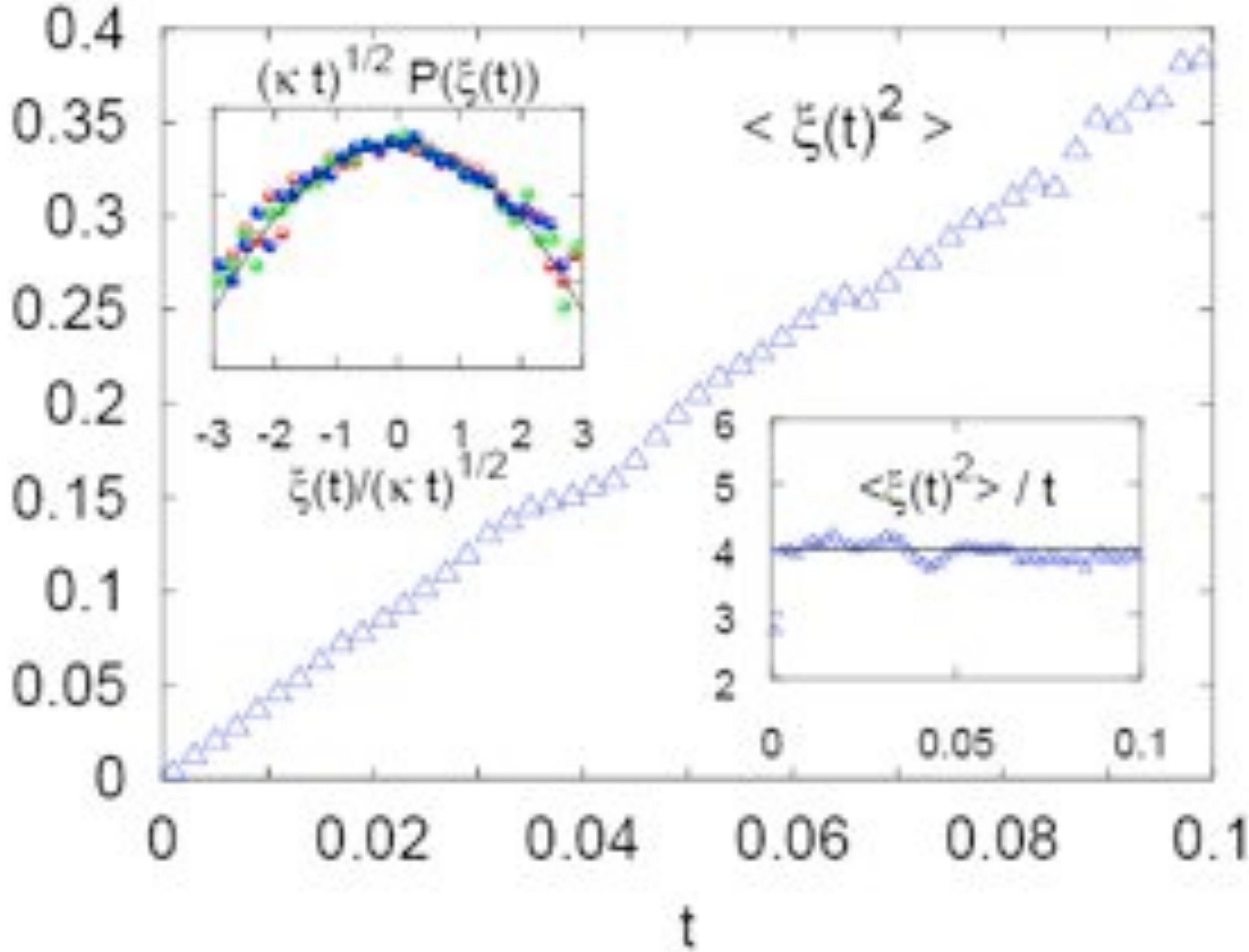
(a)
M
R
$R^{15/8}$
(b)
L
R
$R^{3/2}$
(c)
$n(M, \lambda M)$
M
$M^{-16/15}$
(d)
$n(L, \lambda L)$
L
$L^{-4/3}$
(e)
$n(R, \lambda R)$
R
R^{-2}
(f)
$n(A, \lambda A)$
A
A^{-1}



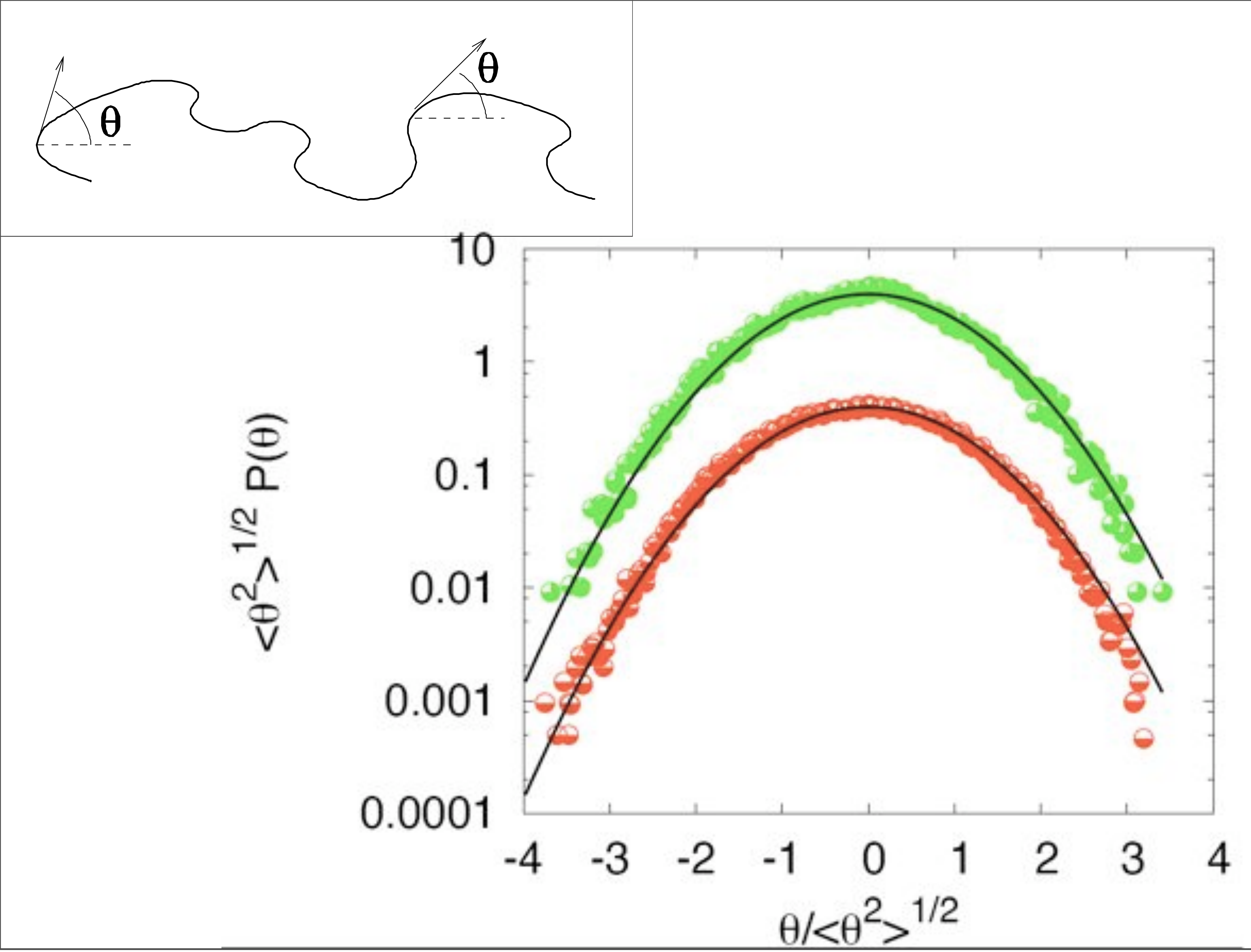
$$\partial_t g_t = 2/\{\varphi'(g_t)[\varphi(g_t) - \xi_t]\}$$



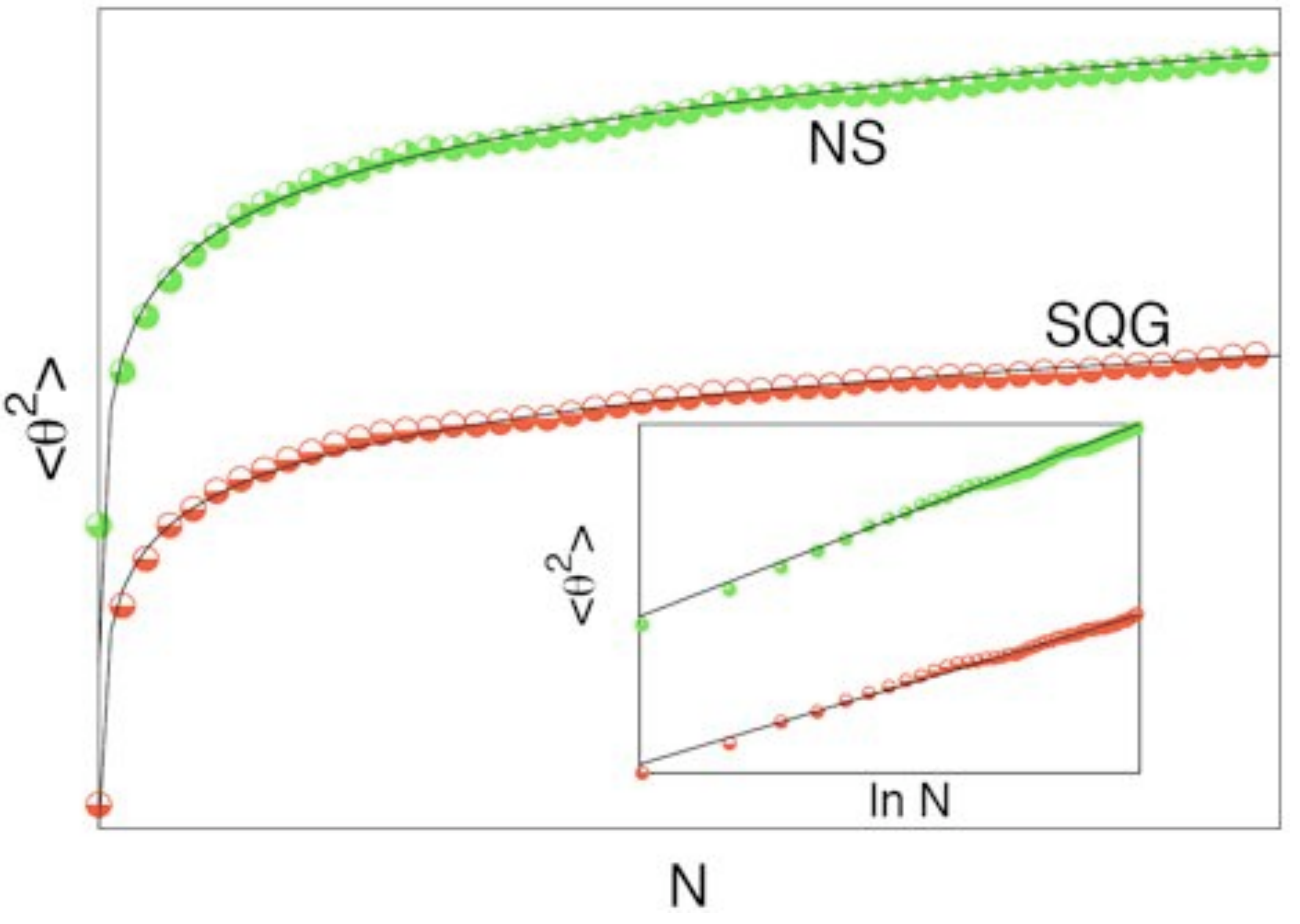
$$\varphi(z) = x_\infty z/(x_\infty - z)$$



< ξ(t)^2 >
(κ t)^1/2 P(ξ(t))
ξ(t)/(κ t)^1/2
<ξ(t)^2> / t
t



θ
θ
10
1
0.1
0.01
0.001
0.0001
-4
-3
-2
-1
0
1
2
3
4
$<\theta^2>^{1/2} P(\theta)$
$\theta/<\theta^2>^{1/2}$

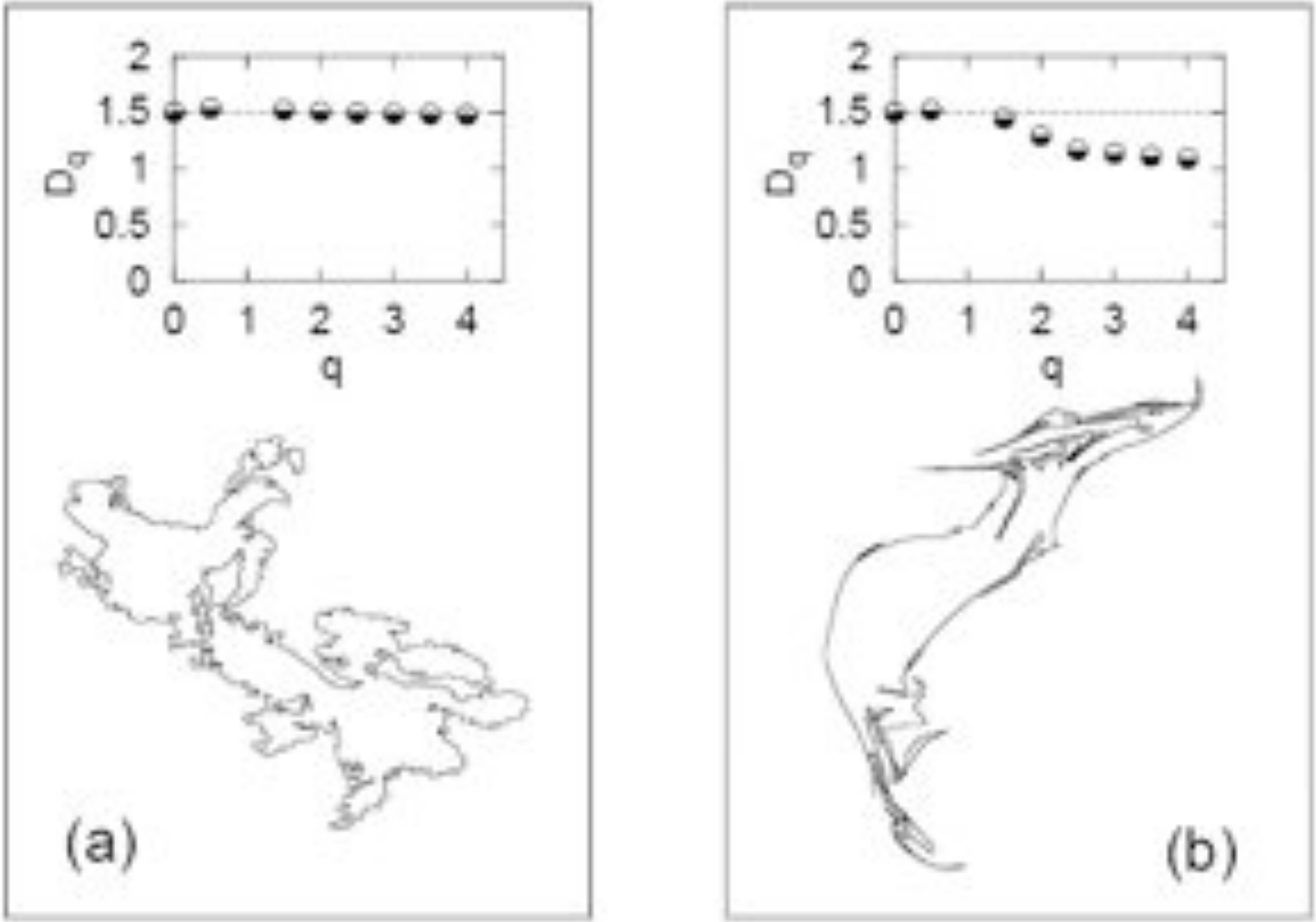

NS
SQG
<θ^2>
N
<θ^2>
ln N

Results:
Within experimental accuracy, isolines of advected quantities behave as SLE in at least two cases of turbulent inverse cascades.

Further questions:
What else in the statistics of turbulence is conformal invariant?
What determines the value of the central charge?
How SLE appears for isolines of non-Gaussian fields?

Different systems producing SLE

- Critical phenomena with local Hamiltonians
- Random walks, non necessarily local
- Nodal lines of wave functions in chaotic systems
- Rocky coastlines
- Spin glasses
- 2d Euler class systems: turbulence



Isolines are scale invariant for the inverse cascade (left panel) but not for the direct cascade (right panel). Both have fractal dimension 3/2.