Blowup or No Blowup? The Interplay between Theory and Numerics

Thomas Y. Hou

Applied and Comput. Mathematis, Caltech

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Computing Euler singularities is an extremely challenging task.

It requires huge computational resource.

Careful resolution study. It is dangerous to interpret the blowup of an under-resolved computation as the blowup of the Euler equations.

Validation check: Is the fitting \( \|\omega\|_{L^\infty} \approx \frac{C}{(T-t)^\alpha} \) asymptotically valid as \( t \to T \) to be used to check if \( \int_0^T \|\omega\|_{L^\infty} \, dt = \infty? \)

Consistency check with other non-blowup criteria. Is there any depletion of vortex stretching? Guidance from the theory is essential.
In 1993 (and 2005), R. Kerr [Phys. Fluids] presented numerical evidence of 3D Euler singularity for two anti-parallel vortex tubes:

- Pseudo-spectral in $x$ and $y$, Chebyshev in $z$ direction;
- Best resolution: $512 \times 256 \times 192$;
- Predicted singularity time $T = 18.7$, but his numerical solutions became under-resolved after $t = 17$; Note that $\Delta = 1.7$ is not small.

- $\|\omega\|_{L^\infty} \approx (T - t)^{-1}$;
- $\|u\|_{L^\infty} \approx (T - t)^{-1/2}$;
- Anisotropic scaling: $(T - t) \times \sqrt{T - t} \times \sqrt{T - t}$;
- Vortex lines: relatively straight, $|\nabla \xi| \approx (T - t)^{-1/2}$;
Figure: Isosurface of peak vorticity at $t = 17$, from R. Kerr, Euler singularities and turbulence, 19th ICTAM Kyoto '96, 1997, pp57-70.
Kerr’s blowup scenario is consistent with the Beale-Kato-Majda (1984) and the Constantin-Fefferman-Majda criteria (1996).


Let $\omega = |\omega| \xi$, no blow-up if

1. (Bounded velocity) $\|u\|_\infty$ is bounded in a $O(1)$ region of large vorticity;

2. (Regular orientedness) $\int_0^t \|\nabla \xi\|_\infty^2 d\tau$ is uniformly bounded;
But it falls into the critical case of Deng-Hou-Yu’s non-blowup criteria.

**Theorem 1** (Deng-Hou-Yu, 2005 and 2006, CPDE)

1. Denote by $L(t)$ the arclength of a vortex line segment $L_t$ around the maximum vorticity. If
   $$\max_{L_t}(|\mathbf{u} \cdot \xi| + |\mathbf{u} \cdot \mathbf{n}|) \leq C_U(T - t)^{-A} \text{ with } A < 1;$$
   $$C_L(T - t)^B \leq L(t) \leq C_0/\max_{L_t}(|\kappa|, |\nabla \cdot \xi|) \text{ with } B < 1 - A;$$

   then the solution of the 3D Euler equations remains regular at $T$.

When $B = 1 - A$, we can exclude blowup if $f(C_U, C_L, C_0) > 0$. For example, $C_L = 1$, $C_0 = 0.1$, $C_U \leq 0.28$ implies no blowup.
We repeat Kerr’s computations using two pseudo-spectral methods.

Four step Runge-Kutta scheme for time integration with adaptive time stepping;

Careful resolution study is performed: $768 \times 512 \times 1536$, $1024 \times 768 \times 2048$ and $1536 \times 1024 \times 3072$.

We compute the solution up to $t = 19$, beyond the alleged singularity time $T = 18.7$ of Kerr.

256 parallel processors with maximal memory consumption 120Gb.

The largest number of grid points is close to 5 billions.
Two spectral methods are used in our computations.

- We use both the $2/3$ dealiasing and a 36-order Fourier smoothing to remove aliasing error;
- The Fourier smoother is shaped as along the $x_j$ direction

$$\rho(2k_j/N_j) \equiv \exp(-36(2k_j/N_j)^{36})$$

where $k_j$ is the wave number ($|k_j| \leq N_j/2$).
Inviscid Burgers equation: spectra comparison with $N = 4096, u_0(x) = \sin(x), T_{\text{shock}} = 1$. 

Figure: Spectra comparison on different resolutions at a sequence of moments. The additional modes kept the Fourier smoothing method higher than the 2/3rd dealiasing method are in fact correct.
Inviscid Burgers equation: the pointwise error comparison with $N = 2048$, $u_0(x) = \sin(x)$, $T_{\text{shock}} = 1$. 

\[\text{pointwise error comparison on 2048 grids, } t=0.9875: \text{blue}(\text{Fourier smoothing}), \text{red}(2/3\text{rd dealiasing})\]
Resolution study of 3D Euler Equations. Enstrophy spectra: $768 \times 512 \times 1024$ vs $1024 \times 768 \times 1536$

Figure: The enstrophy spectra versus wave numbers. The dashed lines and dashed-dotted lines are solutions with $768 \times 512 \times 1024$ using the 2/3 dealiasing rule and the Fourier smoothing, respectively. The times for the spectra lines are at $t = 15, 16, 17, 18, 19$ respectively.

T. Y. Hou, Applied Mathematics, Caltech

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Resolution study of 3D Euler Equations. Energy spectra: $1024 \times 768 \times 2048$ vs $1536 \times 1024 \times 3072$
**Figure:** The 3D vortex tube and axial vorticity on the symmetry plane for initial value.
Figure: The 3D vortex tube and axial vorticity on the symmetry plane when $t = 6$. 
Figure: The local 3D vortex structures and vortex lines around the maximum vorticity at $t = 17$. 
Figure: From: Kerr, Phys. Fluids A 5(7), 1993, pp1725-1746. $t = 15$ (left) and $t = 17$ (right).
Figure: The contour of axial vorticity around the maximum vorticity on the symmetry plane at $t = 15, 17$. 
**Figure:** The contour of axial vorticity around the maximum vorticity on the symmetry plane (the $xz$-plane) at $t = 17.5, 18, 18.5, 19$. 
Figure: The maximum vorticity $\|\omega\|_\infty$ in time, $1024 \times 768 \times 2048$, computed by two spectral methods.
Figure: The inverse of maximum vorticity $\|\omega\|_\infty$ in time using different resolutions.
Dynamic depletion of vortex stretching

Figure: Study of the vortex stretching term in time, resolution $1536 \times 1024 \times 3072$. The fact $|\xi \cdot \nabla u \cdot \omega| \leq c_1 |\omega| \log |\omega|$ plus $\frac{D}{Dt} |\omega| = \xi \cdot \nabla u \cdot \omega$ implies $|\omega|$ bounded by doubly exponential.
Log log plot of maximum vorticity in time

Figure: The plot of $\log \log \|\omega\|_\infty$ vs time, resolution $1536 \times 1024 \times 3072$. 
Double logarithm of peak vorticity in time from Kerr–93 paper.
Figure: Maximum velocity $\|u\|_\infty$ in time using different resolutions.
Recall the local geometric criteria by Deng-Hou-Yu:

1. \( \max_{L_t}(|u \cdot \xi| + |u \cdot n|) \leq C_U (T - t)^{-A} \) for some \( A < 1 \);
2. \( C_L (T - t)^B \leq L(t) \leq C_0 / \max_{L_t}(|\kappa|, |\nabla \cdot \xi|) \) for some \( B < 1 - A \),
then the solution of the 3D Euler equations remains regular up to \( T \).

- Since \( \|u\|_{L^\infty} \) is bounded, we have \( A = 0 \) so our local non-blowup theory applies since \( B = 1/2 < 1 - A \). This rules out a singularity up to \( T = 19 \).
Recall that
\[
\frac{\partial}{\partial t} \omega + (\mathbf{u} \cdot \nabla) \omega = S \cdot \omega, \quad S = \frac{1}{2} (\nabla \mathbf{u} + \nabla^T \mathbf{u}).
\]

Let \( \lambda_1 < \lambda_2 < \lambda_3 \) be the three eigenvalues of \( S \), \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \).

| time   | \( |\omega| \) | \( \lambda_1 \)     | \( \theta_1 \)  | \( \lambda_2 \)   | \( \theta_2 \)  | \( \lambda_3 \)   | \( \theta_3 \)  |
|--------|----------------|---------------------|-----------------|-------------------|-----------------|-------------------|-----------------|
| 16.012 | 5.628          | -1.508              | 89.992          | 0.206             | 0.007           | 1.302             | 89.998          |
| 16.515 | 7.016          | -1.864              | 89.995          | 0.232             | 0.010           | 1.631             | 89.990          |
| 17.013 | 8.910          | -2.322              | 89.998          | 0.254             | 0.006           | 2.066             | 89.993          |
| 17.515 | 11.430         | -2.630              | 89.969          | 0.224             | 0.085           | 2.415             | 89.920          |
| 18.011 | 14.890         | -3.625              | 89.969          | 0.257             | 0.036           | 3.378             | 89.979          |
| 18.516 | 19.130         | -4.501              | 89.966          | 0.246             | 0.036           | 4.274             | 89.984          |
| 19.014 | 23.590         | -5.477              | 89.966          | 0.247             | 0.034           | 5.258             | 89.994          |

Table: The alignment of the vorticity vector and the eigenvectors of \( S \) around the point of maximum vorticity with resolution 1536 \( \times \) 1024 \( \times \) 3072. Here, \( \theta_i \) is the angle between the \( i \)-th eigenvector of \( S \) and the vorticity vector.
We have also repeated Pelz’s computations, and found no evidence of a finite time singularity.

Pelz’s filament model indeed leads to a finite time blowup [PRE, 97]. But when we use the same high symmetry initial condition to solve the full 3D Euler equations, the solution remains regular.

Boratav and Pelz’s Navier-Stokes computations [Phys Fluid, 94] suggested a potential singularity around $t = 2.06$ as $Re \to \infty$.

Our resolution study shows that their computations are resolved only up to $t = 1.6$ when the growth is only exponential in time. The rapid growth around $t = 2.06$ seems due to under-resolution.

We have used two codes to compute the high symmetry solution, one code built in the high symmetry explicitly, the other did not. The symmetry is preserved by the second code to many digits.
Maximum vorticity of the high symmetry data in time, one code built in high symmetry explicitly, the other did not.

![Graph showing maximum vorticity as a function of time with two codes compared: symmetric code (solid line) and non-symmetric code (dashed line).](image)
Concluding Remarks

- Our analysis and computations reveal a subtle dynamic depletion of vortex stretching. Sufficient numerical resolution is essential in capturing this dynamic depletion.

- Our computations show that the velocity is bounded and that $\|\xi \cdot \nabla u \cdot \omega\|_{L^\infty} = O(\|\omega\|_{L^\infty} \log(\|\omega\|_{L^\infty}))$, instead of $\|\omega\|_{L^\infty}^2$.

- It is natural to ask what is the driving mechanism for this dynamic depletion of vortex stretching? Is this scaling generic?

- The geometric regularity of local vortex lines and the anisotropic scaling of the support of maximum vorticity seem to play an important role in the dynamic depletion of vortex stretching.

- New analytic tools that exploit the local geometric structure of the solution near a potential singularity are needed.


