A Geometrical Study of 3D Incompressible Euler Flows with Clebsch potentials

Inviscid longevity and Kolmogorov Spectrum

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21 June 2007 @ Euler 250, Aussois
KE: regularity, Euler equations, Onsager conjecture, numerical simulation

Motivation

Singularity formation in Euler flows ?

Relevance to turbulence ?

$$\epsilon = \nu \left\langle |\boldsymbol{\omega}|^2 \right\rangle$$

$$\nu \equiv 0 \ \textbf{vs.} \ \nu \to 0$$

Onsager conjecture for $\nu = 0$ (Eyink, this morning)

Blowup may drop energy;

Use Euler eqs. to study fully-developed turbulence

Inviscid and inviscid limit behaviour(Two time scales)
t_***=rapid growth in vorticity, possible singularity ?** $(\nu = 0)$
T_***=total enstrophy peaked, followed by K41** $(\nu > 0)$

Taylor-Green vortex $t_* \approx 5(?)$, $T_* \approx 9$ **(Brachet et al.)**
Kida high-symmetric flow $t_* \approx 2(?)$, $T_* \approx 4$ **(Pelz et al.)**
A simple flow with Clebsch potentials $T_* \approx 8$

Go for geometrically the simplest flows
Characterise them as thoroughly as possible

Outline

0. Review of Clebsch's works
1. Mathematical Formulation: Clebsch Potentials
2. Condition for geometrical non-degeneracy
3. Preliminary results by numerics
4. Summary

0. Clebsch’s papers

“Uber eine allgemeine Transformation der hydrodynamischen Gleichungen”
J Reine Angew Math 54(1857)293–313.

“Uber die Integration der hydrodynamischen Gleichungen”
J Reine Angew Math 56(1859)1–10.

Note: “Uber integrale der hydrodynamischen Gleichungen, welche den wirbelbewegungen entsprechen”
H. Helmholtz, J Reine Angew Math 55(1858)25–55

“Report on Recent Progress in Hydrodynamics.–Part I”
W.M. Hicks, British Association (1881).

Clebsch(1857): Variational principle for stationary case

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + \ldots + u_n \frac{\partial u_1}{\partial x_n} = -\frac{\partial p}{\partial x_1}, \textbf{etc}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \ldots + \frac{\partial u_n}{\partial x_n} = 0$$

Consider $a_0(x_1, \ldots, x_n), \ldots, a_{n-1}(x_1, \ldots, x_n)$

$$R = \begin{vmatrix} \frac{\partial a_0}{\partial x_1} & \frac{\partial a_0}{\partial x_2} & \cdots & \frac{\partial a_0}{\partial x_n} \\ \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \cdots & \frac{\partial a_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial a_{n-1}}{\partial x_1} & \frac{\partial a_{n-1}}{\partial x_2} & \cdots & \frac{\partial a_{n-1}}{\partial x_n} \end{vmatrix}$$

$$R = \Delta_1 \frac{\partial a_0}{\partial x_1} + \Delta_2 \frac{\partial a_0}{\partial x_2} + \ldots + \Delta_n \frac{\partial a_0}{\partial x_n}$$

$$\frac{\partial \Delta_1}{\partial x_1} + \frac{\partial \Delta_2}{\partial x_2} + \ldots + \frac{\partial \Delta_n}{\partial x_n} = 0$$

$$\Delta_i = \Delta_i(a_1, a_2, \ldots, a_{n-1}) \to u_i$$

Δ lies on $\Pi(a_1, a_2, \ldots, a_{n-1})$=const.

$$\frac{\partial \Delta_1}{\partial t} + A_1 \frac{\partial a_1}{\partial x_1} + A_2 \frac{\partial a_2}{\partial x_1} + \ldots + A_{n-1} \frac{\partial a_{n-1}}{\partial x_1} = -\frac{\partial}{\partial x_1} \left(p + \frac{|\boldsymbol{u}|^2}{2} \right)$$

$$A_1 \equiv \frac{\partial \Pi}{\partial a_1}, A_2 \equiv \frac{\partial \Pi}{\partial a_2}, \ldots, A_{n-1} \equiv \frac{\partial \Pi}{\partial a_{n-1}}$$

Stationary case $-\left(p + \frac{|\boldsymbol{u}|^2}{2} \right) = \Pi(a_1, a_2, \ldots, a_{n-1})$

Ex. $n = 3,\ \boldsymbol{u} = \nabla a_1 \times \nabla a_2 = \nabla \times (a_1 \nabla a_2)$

In particular, $a_1 = \psi(x_1, x_2), a_2 = x_3$

$$\frac{d}{d\psi}\left(p + \frac{|\boldsymbol{u}|^2}{2}\right) = -\omega, \quad \omega = \omega(\psi)$$

Clebsch(1859): Variational principle for non-stationary case

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x_0} + u_1 \frac{\partial u_0}{\partial x_1} + \ldots + u_{2n} \frac{\partial u_0}{\partial x_{2n}} = -\frac{\partial p}{\partial x_0},$$

$$\frac{\partial u_0}{\partial x_0} + \frac{\partial u_1}{\partial x_1} + \ldots + \frac{\partial u_{2n}}{\partial x_{2n}} = 0$$

$$u_k = \frac{\partial \phi_0}{\partial x_k} + m_1 \frac{\partial \phi_1}{\partial x_k} + \ldots + m_n \frac{\partial \phi_n}{\partial x_k}$$

$$-\delta \Pi \equiv -\delta \left(p + \frac{|\boldsymbol{u}|^2}{2} + \frac{\partial \phi}{\partial t} + m_j \frac{\partial \phi_j}{\partial t} \right) = \frac{D m_j}{Dt} \delta \phi_j - \frac{D \phi_j}{Dt} \delta m_j$$

$$\frac{D m_j}{Dt} = \frac{\delta \Pi}{\delta \phi_j}, \; \frac{D \phi_j}{Dt} = -\frac{\delta \Pi}{\delta m_j}$$

1. Mathematical Formulation

Euler equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p, \quad \nabla \cdot \boldsymbol{u} = 0$$

$$\int_0^T \max_{\boldsymbol{x}} |\boldsymbol{\omega}(\boldsymbol{x}, t)| dt < \infty$$

Beale-Kato-Majda (1984)

$$\int_0^T \max_{\boldsymbol{x}} |\nabla f(\boldsymbol{x}, t)|^2 dt < \infty$$

advected scalar f, Constantin (2001)

Clebsch potential $\boldsymbol{u} = f\nabla g - \nabla\phi$

$$\boldsymbol{\omega} = \nabla f \times \nabla g$$

Take $\dfrac{Df}{Dt} = \dfrac{Dg}{Dt} = 0$ **for simplicity**

***Kinematics: Vector analysis (Frobenius's condition)**

$$\boldsymbol{\gamma} = f\nabla g \ \ \textbf{globally} \Leftrightarrow \boldsymbol{\gamma}\cdot\nabla\times\boldsymbol{\gamma} \equiv 0$$

***Dynamics** $\boldsymbol{\gamma} = \boldsymbol{u} + \nabla\phi$

$$\frac{D}{Dt}\boldsymbol{\gamma}\cdot\nabla\times\boldsymbol{\gamma} = 0$$

2. Condition for geometrical non-degeneracy

$$\boldsymbol{\omega} = \nabla f \times \nabla g$$

Minimum rates for possible blow-up

$$\max|\boldsymbol{\omega}| = O\left(\frac{1}{T-t}\right) \quad \textbf{BKM 1984}$$

$$\max|\nabla f|, \max|\nabla g| = O\left(\frac{1}{\sqrt{T-t}}\right) \quad \textbf{Constantin 2001}$$

If ∇f tends to be colinear with ∇g we would have a contradiction.

3. Preliminary numerical results

Initial Conditions

(1) A simple flow with Clebsch potentials

(2) Kida's high-symmetric flow

(3) Taylor-Green(-Orr) vortex

Solve simultaneously by pseudo-spectral method

(resolution $2/3-$ **dealiased** 256^3**)**

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p, \quad \nabla \cdot \boldsymbol{u} = 0$$

$$\frac{Df}{Dt} = \frac{Dg}{Dt} = 0$$

Check $\boldsymbol{\omega} = \nabla f \times \nabla g$ **pointwise**

(1) A simple initial condition

$$f = \sin x + \sin y + \sin z,\ \ g = \cos x + \cos y + \cos z$$

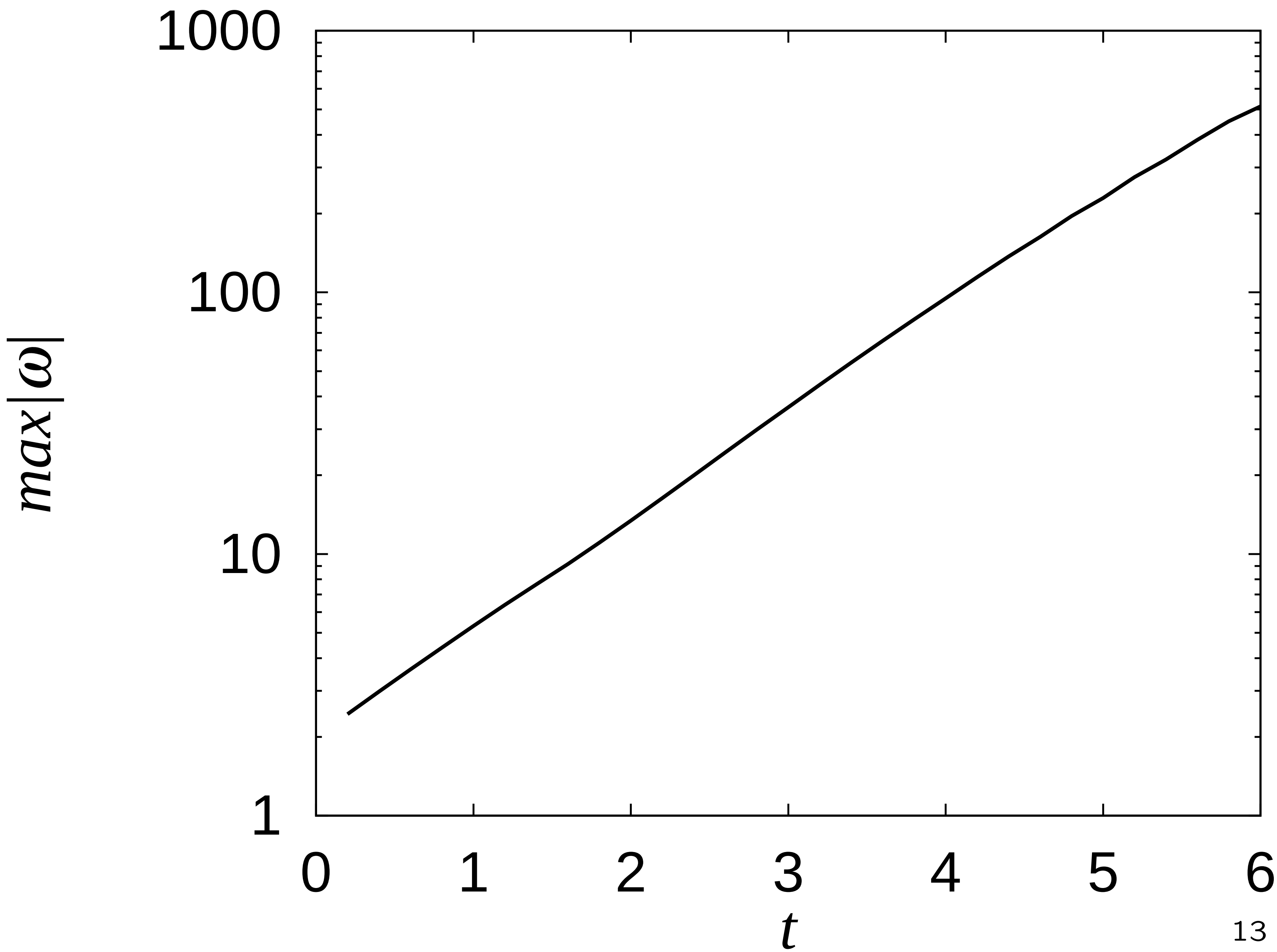
$$\boldsymbol{\omega} = \nabla f \times \nabla g = [\sin(y-z), \sin(z-x), \sin(x-y)]$$

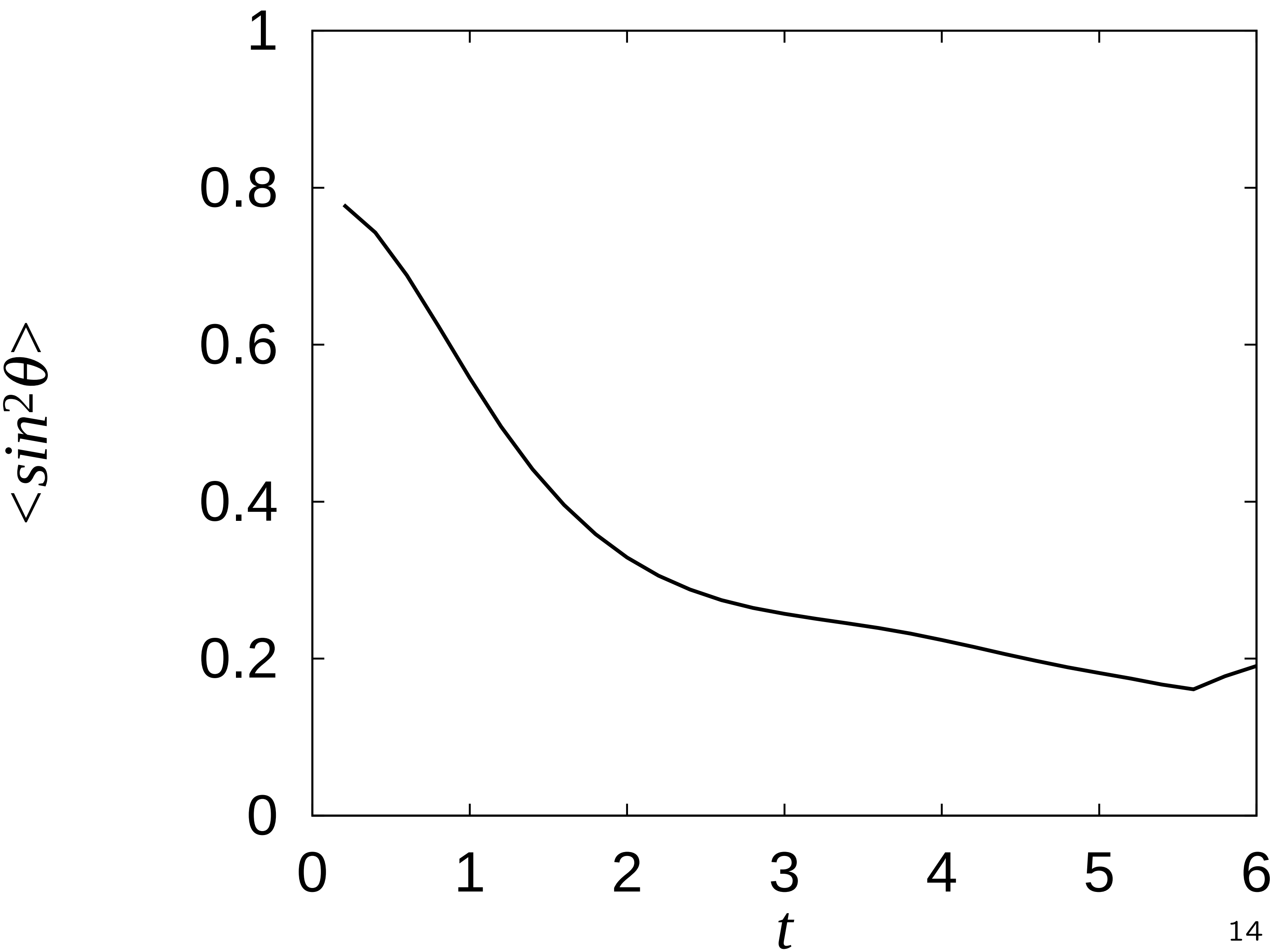
$$\boldsymbol{u} = \left[-\frac{1}{2}\left(\cos(x-y) + \cos(x-z) + 1\right),\right.$$

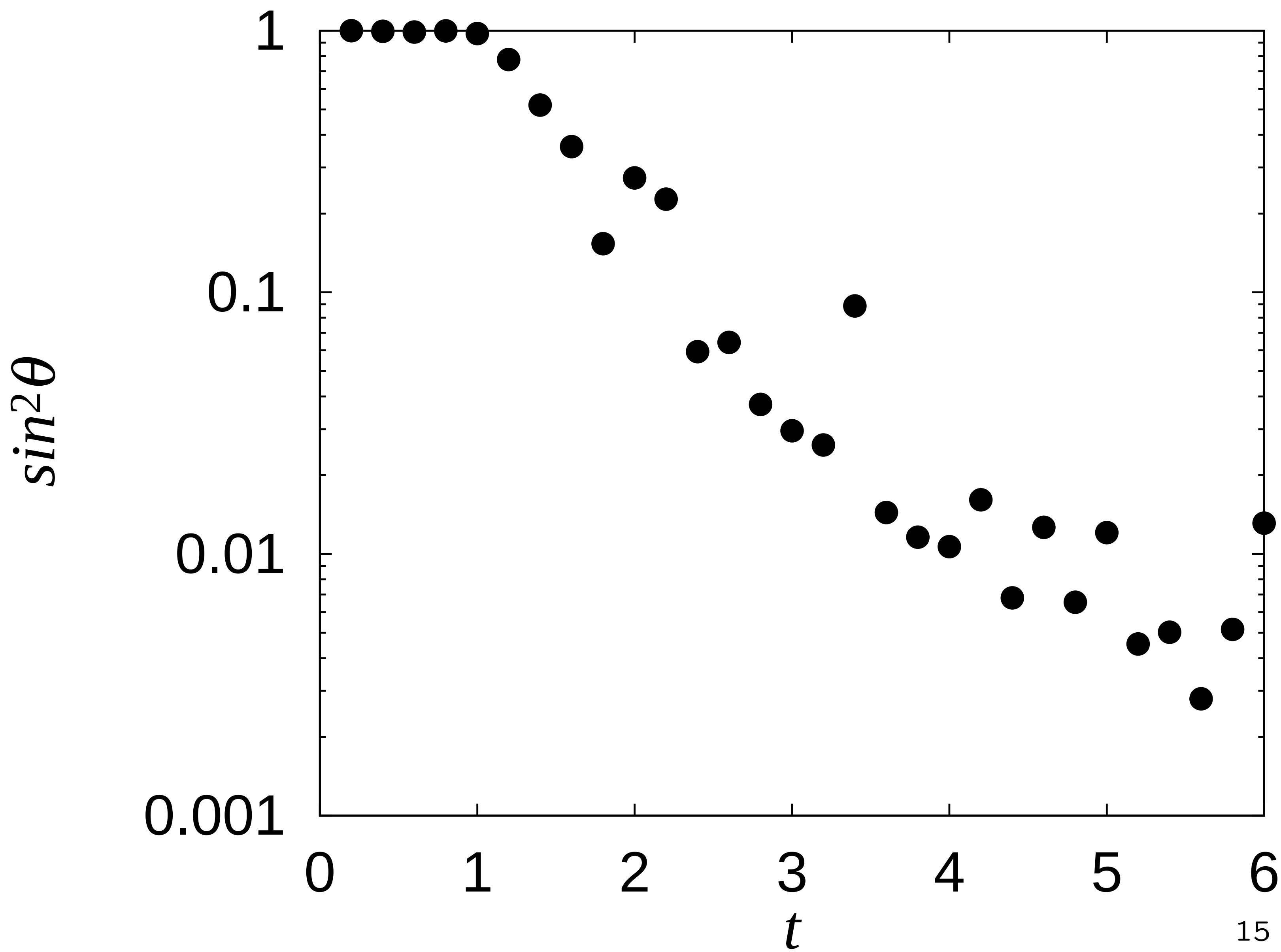
$$\left. -\frac{1}{2}\left(\cos(y-z) + \cos(y-x) + 1\right), -\frac{1}{2}\left(\cos(z-x) + \cos(z-y) + 1\right)\right]$$

$\max|\boldsymbol{\omega}|$, $\left\langle \sin^2\theta \right\rangle$, $\sin^2\theta$

$$\sin^2\theta \equiv \frac{(\nabla f \times \nabla g)^2}{|\nabla f|^2 |\nabla g|^2}$$


1000
100
10
1
0
1
2
3
4
5
6
t
max|ω|


$\langle sin^2\theta \rangle$
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
t


1
0.1
0.01
0.001
$sin^2\theta$
0
1
2
3
4
5
6
t

Navier-Stokes equations

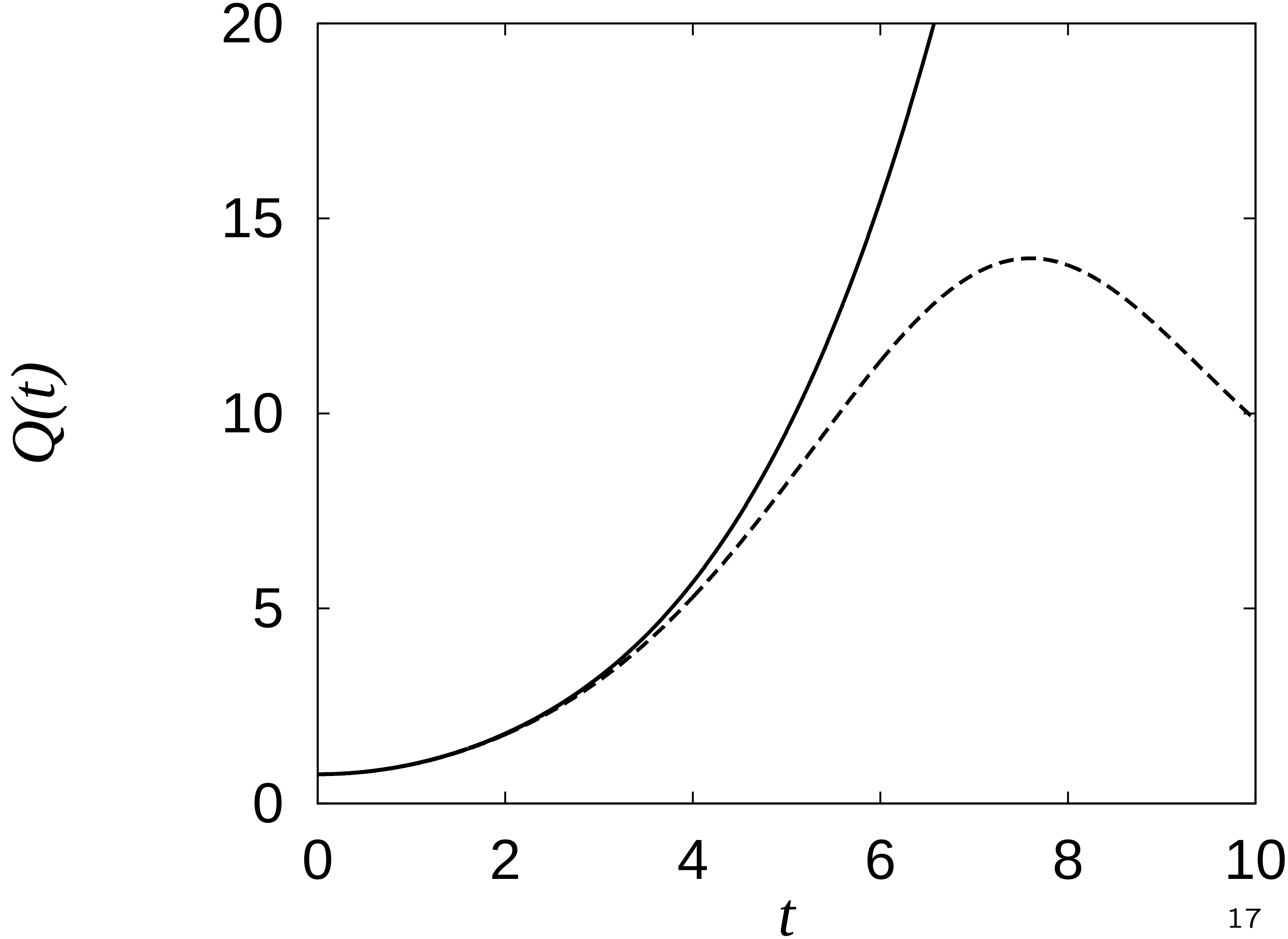
$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p + \nu \triangle \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0$$

$$Q(t) = \left\langle \frac{|\boldsymbol{\omega}|^2}{2} \right\rangle$$

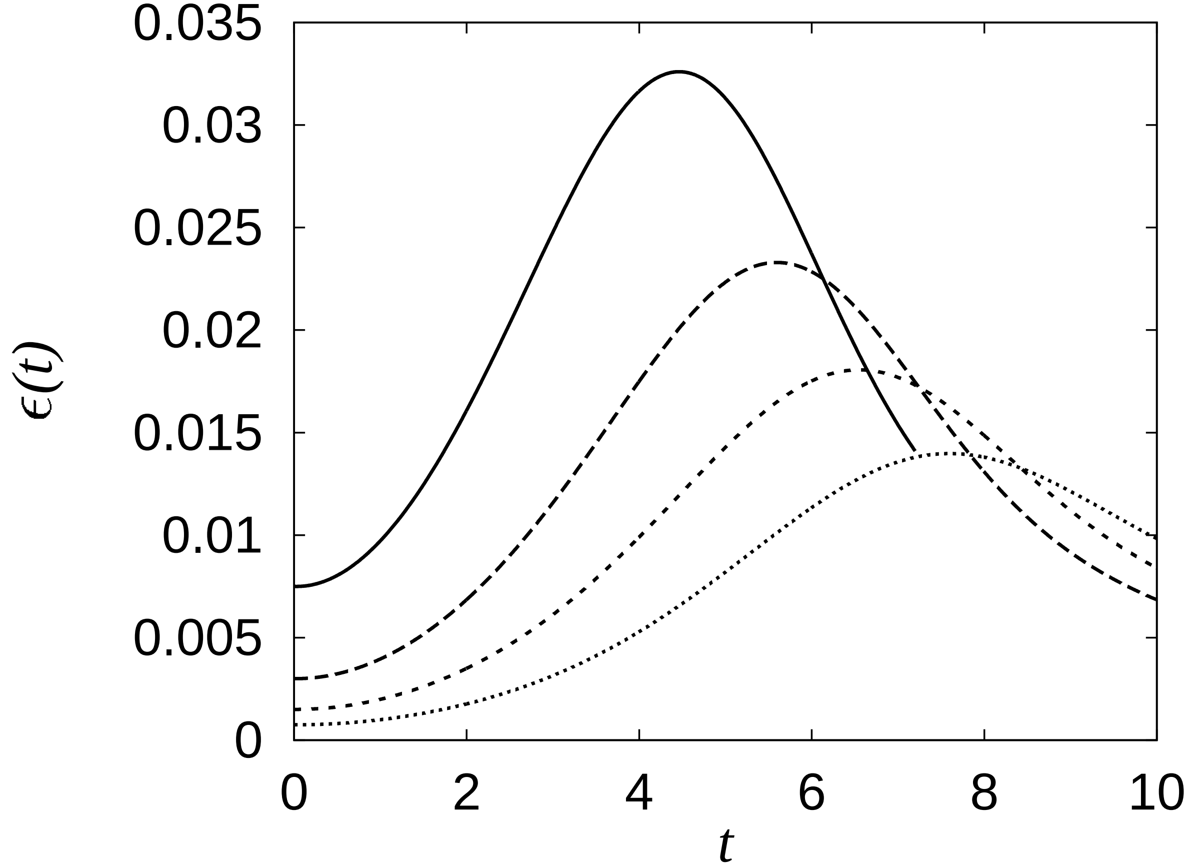
$$\epsilon(t) = \nu \left\langle |\boldsymbol{\omega}|^2 \right\rangle$$

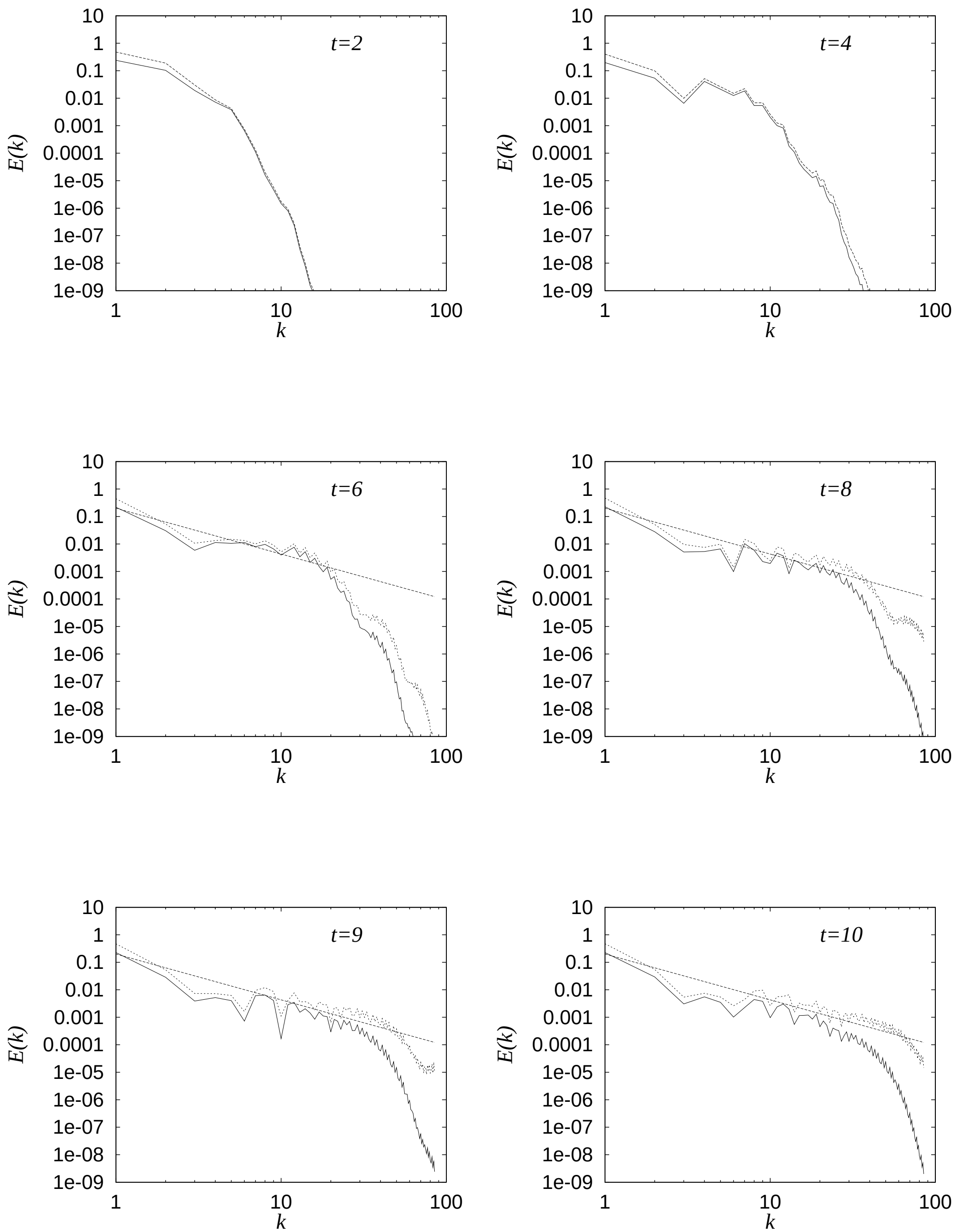
kinematic viscosity

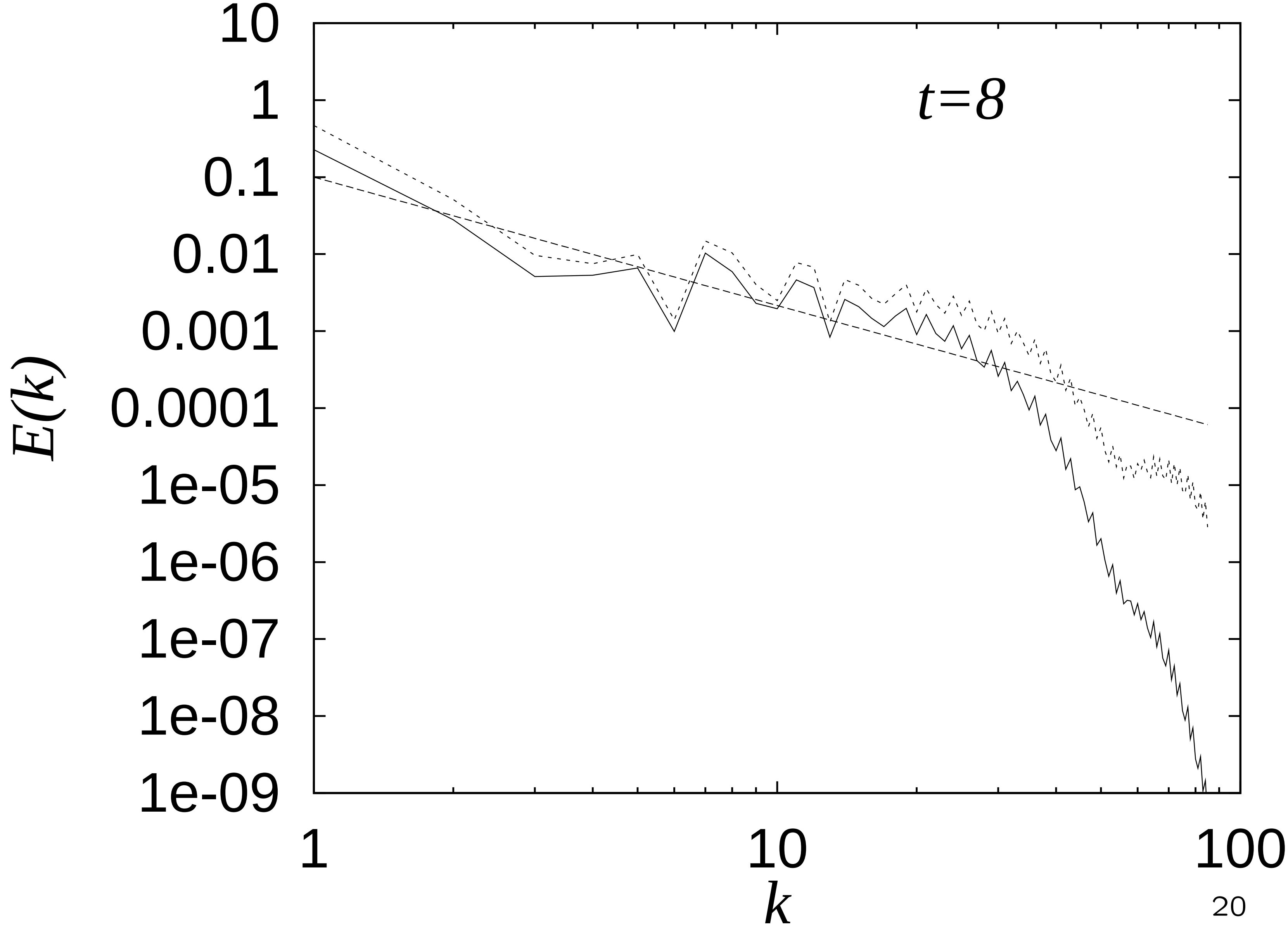
$$\nu = 5.0,\ 2.0,\ 1.0,\ 0.5 \times 10^{-3}$$

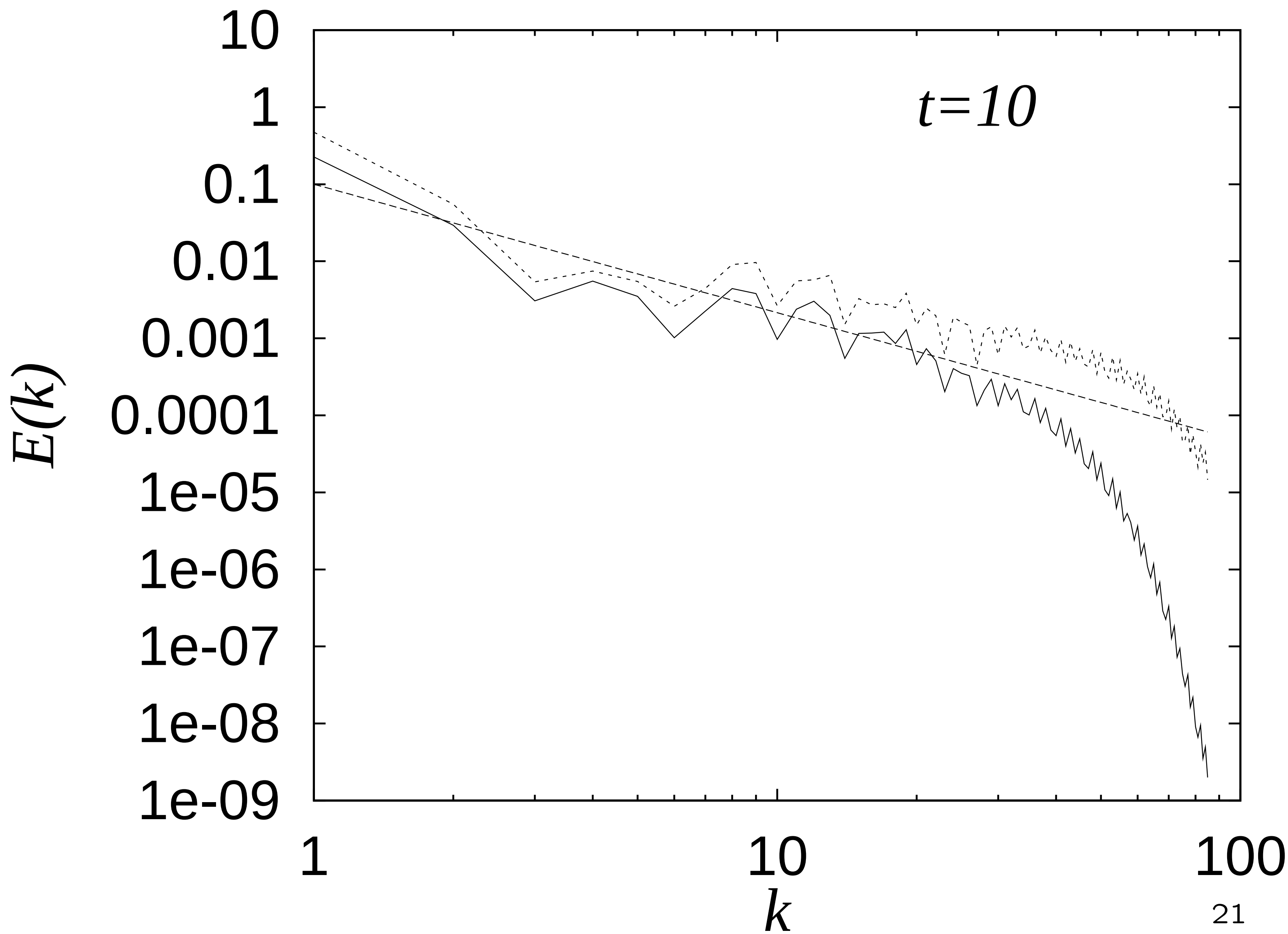

20
15
10
5
0
0
2
4
6
8
10
t
$Q(t)$

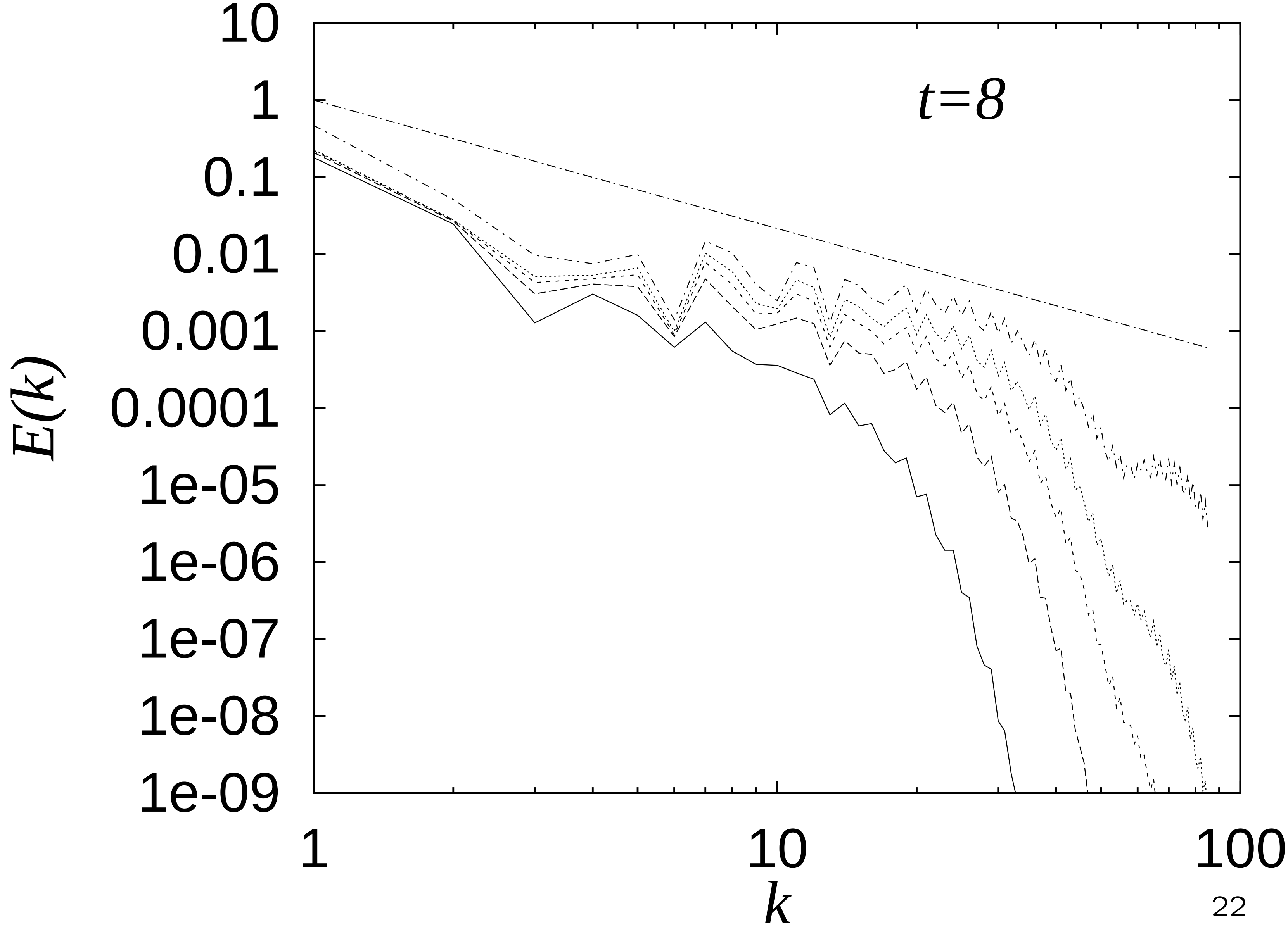
$T_* \propto \nu^{-\alpha},\ \ \alpha \approx 0.2,\ \epsilon_{\mathrm{max}} \propto \nu^{\beta},\ \ \beta \approx 0.4$




t=2
t=4
t=6
t=8
t=9
t=10
E(k)
k
10
1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
1
10
100


t=8
E(k)
k
10
1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
1
10
100


t=10
E(k)
k
10
1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
1
10
100



t=8
10
1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
E(k)
1
10
100
k

Features of this simple flow

*$t_* \approx T_*$

***Max time of** $\epsilon(t)$ **recedes to** ∞ **as** $\nu \to 0$

*$\epsilon(t)$ **at its max** $\to 0$ **as** $\nu \to 0$

***With** $\nu > 0$, $E(k) \propto k^{-5/3}$ **after** $Q(t)$ **reaches a max**

***With** $\nu = 0$, $\max|\boldsymbol{\omega}| \propto \exp(ct)$

***With** $\nu = 0$, **apparently consistent with** $E(k) \propto k^{-5/3}$
(to be checked with higher resolution)

Not to be confused with 'warm cascade'
= turbulence + heat bath ($E(k) \propto k^2$**),**
Nazarenko-Connaghton(2004), Brachet et al(2005)

A long-lived Euler flow sustains Kolmogorov Spectrum ?

Comparison theorem: Constantin(1986)

For fixed t, $E_\nu(k) \to E_0(k)$ **as** $\nu \to 0$

not necessarily $E_0(k) \propto k^{-5/3} \quad \because \; t \ll T_*(\nu)$

Can we find $E(k) \propto k^{-5/3}$ **at** $t = T_*(\nu)$ **as** $\nu \to 0$**,**
even if $\epsilon(t) \to 0$ **?**

(2) Kida's high-symmetric flows

$(0 \leq x, y, z < \pi/2)$ ($\boldsymbol{\gamma} = \boldsymbol{u}$ **at** $t = 0$)

$$\boldsymbol{u} = \begin{pmatrix} \sin x(\cos 3y \sin z - \cos y \sin 3z) \\ \sin y(\cos 3z \sin x - \cos z \sin 3x) \\ \sin z(\cos 3x \sin y - \cos x \sin 3y) \end{pmatrix}$$

$$\boldsymbol{\omega} = \begin{pmatrix} -2\cos 3x \sin x \sin z + 3\cos x(\sin 3y \sin z + \sin y \sin 3z) \\ -2\cos 3y \sin y \sin x + 3\cos y(\sin 3z \sin x + \sin z \sin 3x) \\ -2\cos 3z \sin z \sin y + 3\cos z(\sin 3x \sin y + \sin x \sin 3y) \end{pmatrix}$$

$$\boldsymbol{u} \cdot \boldsymbol{\omega} \equiv 0$$

Clebsch potentials for Kida flow
we may choose

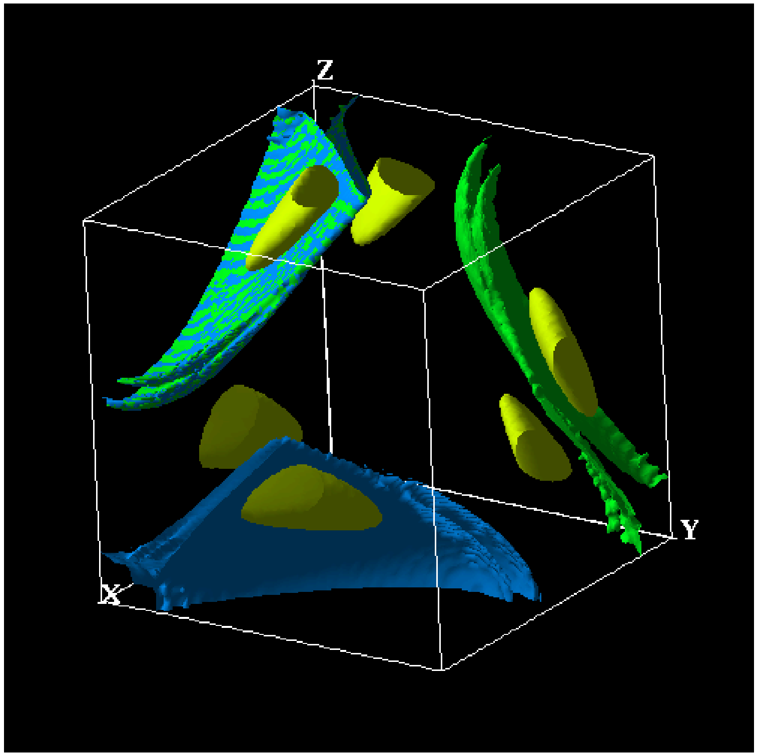
$$f = 2\frac{(\cos x)^{3/2}(\cos^2 y - \cos^2 z)}{(\cos y \cos x)^{1/2}} \approx \frac{z^2 - y^2}{2}$$

$$g = 2\frac{(\cos y)^{3/2}(\cos^2 z - \cos^2 x)}{(\cos x \cos y)^{1/2}} \approx \frac{x^2 - z^2}{2}$$

cf.

$$\boldsymbol{\gamma} \cdot \boldsymbol{\omega} = \boldsymbol{u} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \phi$$

(Need a workaround against apparent singularities in Clebsch potentials)



$t = 0.6$

5. Summary

***Yet another depletion mechanism:**
colinearity of ∇f and $\nabla g \rightarrow$ exponential growth

***Identified longevity of an inviscid flow with mild energy transfer**

***Demonstrating coexistence of smooth Euler evolution with the Kolmogorov scaling for the slightly viscous case**

***As long as this flow is concerned, the physical motivation for suspecting singularity is lost**

***Offers a hands-on example for 'turbulence without singularity'**

Outlook
***Can the inviscid solution yield $E(k) \propto k^{-5/3}$?**

***Are solutions with weak energy transfer sporadic ?**

***What are the implications for more general flows with non-vanishing $\epsilon(t)$?**
Special $\subset$ Clebsch $\subset$ General Incompressible

***Two cases**
(A)All Euler flows remain regular(quantitative difference)
Regularity trivialises Onsager's approach ? Can the Euler eqs. characterise developed turbulence?

(B)Some Euler flows go singular(qualitative difference)
Onsager conjecture may work literally

References (not exhaustive)

Duhem, *Sur les équations de h'hydrodynamique. Commentaire a un mémoire de Clebsch*, **Ann Toulouse(1901)**

Serrin, *Mathematical Principles of Classical Fluid Mechanics*, **Handbuch der Physik(1959)**

Benjamin, *Impulse, flow force and variational principles*, **IMA J Appl Math 32(1984)**

Kambe, *Gauge principle and variational formulation for ideal fluids with reference to translational symmetry*, **Fluid Dyn Res 39(2007)**