

FROM NEWTON'S MECHANICS TO EULER'S EQUATIONS

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The Euler equations of hydrodynamics, which appeared in their present form in the 1750s, did not emerge in the middle of a desert. We shall see in particular how the Bernoullis contributed much to the transmutation of hydrostatics into hydrodynamics, how d'Alembert was the first to describe fluid motion using partial differential equations and a general principle linking statics and dynamics, and how Euler developed the modern concept of internal pressure field which allowed him to apply Newton's second law to infinitesimal elements of the fluid.¹

Quelques sublimes que soient les recherches sur les fluides, dont nous sommes redevables à Mrs. *Bernoullis*, *Clairaut*, & *d'Alembert*, elles découlent si naturellement de mes deux formules générales : qu'on ne scauroit assés admirer cet accord de leurs profondes méditations avec la simplicité des principes, d'où j'ai tiré mes deux équations, & auxquels j'ai été conduit immédiatement par les premiers axiomes de la Mécanique.² (Leonhard Euler 1755)

I. INTRODUCTION

Leonhard Euler had a strong interest in fluid dynamics and related subjects during all his adult life. In 1827, at age twenty, he published an important paper on the theory of sound. There he gave a quantitative theory of the oscillations of the column of air in a flute or similar instruments. On a slate found after his death on 7 September 1783 he developed a theory of aerostatic balloons, having just learned about the first manned ascent of a balloon designed by the brothers Montgolfier. Altogether he published more than forty papers or books devoted to fluid dynamics and applications. After his arrival in Saint-Petersburg in 1727, and perhaps before, Euler was planning a treatise on fluid mechanics based on the principle of live forces. He recognized the simi-

larity of his project with Daniel Bernoulli's and left the field open to this elder friend. During the fourteen years of his first Petersburg stay, Euler was actively involved in establishing the theoretical foundations of naval science, thereby contributing to the ongoing effort of the Russian state in developing a modern and powerful fleet. His *Scientia Navalis*, completed by 1738 and published in 1749, contained a clear formulation of hydrostatic laws and their application to the problem of ship stability. It also involved a few Newtonian considerations on ship resistance. Soon after his move to Berlin in 1741, he edited the German translation of Benjamin Robins's *New Principles of Gunnery*, as a consequence of Frederick II's strong interest in the science of artillery. Published in 1745, this edition included much innovative commentary on the problem of the resistance of the air to the motion of projectiles, especially regarding the effects of high speed and cavitation.³

Today's fluid dynamics cannot be conceived without the fundamental basis of Euler's equations, as they appear in "Principes généraux du mouvement des fluides", presented to the Académie Royale des Sciences et Belles-Lettres (Berlin) on 4 September 1755 and published in 1757. In Euler's own notation they read:

$$\left. \begin{aligned} \left(\frac{dq}{dt} \right) + \left(\frac{d \cdot qu}{dx} \right) + \left(\frac{d \cdot qv}{dy} \right) + \left(\frac{d \cdot qw}{dz} \right) &= 0 \\ P - \frac{1}{q} \left(\frac{dp}{dx} \right) &= \left(\frac{du}{dt} \right) + u \left(\frac{du}{dx} \right) + v \left(\frac{du}{dy} \right) + w \left(\frac{du}{dz} \right) \\ Q - \frac{1}{q} \left(\frac{dp}{dy} \right) &= \left(\frac{dv}{dt} \right) + u \left(\frac{dv}{dx} \right) + v \left(\frac{dv}{dy} \right) + w \left(\frac{dv}{dz} \right) \\ R - \frac{1}{q} \left(\frac{dp}{dz} \right) &= \left(\frac{dw}{dt} \right) + u \left(\frac{dw}{dx} \right) + v \left(\frac{dw}{dy} \right) + w \left(\frac{dw}{dz} \right). \end{aligned} \right\} \quad (1)$$

¹The present article includes large sections of Chapter 1 of Darrigol 2005, thanks to the kind permission of Oxford University Press.

We mention that one of the authors (OD) is a theoretical physicist by early training who became a historian of science some twenty years ago, while the other one (UF) is a fluid dynamicist interested in Euler's equations since the seventies.

² Euler 1755c : 316[original publication page]/92[omnia page] : However sublime the researches on fluids that we owe to Messrs *Bernoullis*, *Clairaut*, and *d'Alembert* may be, they derive so naturally from my two general formulas that one could not cease to admire this agreement of their profound meditations with the simplicity of the principles from which I have drawn my two equations and to which I have been immediately driven by the first axioms of Mechanics.

³ Euler 1727, [1784] (balloons), 1745, 1749. For general biography, cf. Youschkevitch 1971; Knobloch 2008 and references therein. On Euler and hydraulics, cf. Mikhailov 1983. On sound, cf. Truesdell 1955: XXIV–XXIX. On the early treatise on fluids, cf. Mikhailov 1999, and pp. 61–62, 80 in Euler 1998. On naval science, cf. Nowacki 2006; Truesdell 1954: XVII–XVIII, 1983. On gunnery, cf. Truesdell 1954: XXVIII–XLI.

Here, P , Q and R are the components of an external force, such as gravity. The modern reader with no special training in history of science will nevertheless recognize these equations and be barely distracted by the use of q instead of ρ for the density, of $\left(\frac{du}{dx}\right)$ instead of $\frac{\partial u}{\partial x}$ and of $d \cdot qu$ instead of $\partial(qu)$.⁴

Euler's three memoirs on fluid dynamics written in 1755 contain of course much more than these equations. They are immediately intelligible to the modern reader, the arguments being strikingly close to those given in modern treatises. They mark the emergence of a new style of mathematical physics in which fundamental equations take the place of fundamental principles formulated in ordinary or geometrical language. Euler's equations are also the first instance of a *nonlinear* field theory and remain to this day shrouded in mystery, contrary for example to the heat equation introduced by Fourier in 1807 and the Maxwell equations, discovered in 1862.

Our main goal is to trace the development and maturation of physical and mathematical concepts, such as internal pressure, which eventually enabled Euler to produce his memoirs of the 1750s.⁵ The emergence of Euler's equations was the result of several decades of intense work involving such great figures as Isaac Newton, Alexis Clairaut, Johann and Daniel Bernoulli, Jean le Rond d'Alembert ... and Euler himself. It is thus also our goal to help the reader to see how such early work, which is frequently difficult because it is not couched in modern scientific language, connects with Euler's maturing views on continuum mechanics and his papers of the 1750s.

Section II is devoted to the first applications of Newtonian mechanics to fluid flow, from Newton to the Bernoullis. Whereas Isaac Newton treated a few particular problems with heteroclitic and ad hoc methods, Daniel and Johann Bernoulli managed to solve a large class of problems through a uniform dynamical method. Section III shows how Jean le Rond d'Alembert's own dynamical method and mathematical creativity permitted a great extension of the investigated class of flows. Despite its now antiquated formulation, his theory had many of the key concepts of the modern theory of incompressible flow. In Section IV we discuss Euler's memoirs of the 1750s. Finally, a few conclusions are presented in Section V.

Another paper in these Proceedings focuses on Euler's 1745 third remark (Theorem 1) à propos Robins's Gunnery. This remark, which constitutes actually a standalone paper of eleven pages on the problem of steady flow around a solid body, is at the crossroads of eighteenth-century fluid dynamics: it uses many ideas of the Bernoullis to write the equations in local coordinates

and has been viewed, correctly or not, as a precursor of d'Alembert's derivation of the paradox of vanishing resistance (drag) for ideal flow.⁶

II. FROM NEWTON TO THE BERNOULLIS

Newton's Principia

Through the eighteenth century, the main contexts for studies of fluid motion were water supply, water-wheels, navigation, wind-mills, artillery, sound propagation, and Descartes's vortex theory. The most discussed questions were the efflux of water through the short outlet of a vessel, the impact of a water vein over a solid plane, and fluid resistance for ships and bullets. Because of its practical importance and of its analogy with Galilean free-fall, the problem of efflux got special attention from a few pioneers of Galilean mechanics. In 1644, Evangelista Torricelli gave the law for the velocity of the escaping fluid as a function of the height of the water level; in the last quarter of the same century, Edme Mariotte, Christiaan Huygens, and Isaac Newton tried to improve the experimental and theoretical foundations of this law.⁷

More originally, Newton devoted a large section of his *Principia* to the problem of fluid resistance, mainly to disprove the Cartesian theory of planetary motion. One of his results, the proportionality of inertial resistance to the square of the velocity of the moving body, only depended on a similarity argument. His more refined results required some drastically simplified model of the fluid and its motion. In one model, he treated the fluid as a set of isolated particles individually impacting the head of the moving body; in another he preserved the continuity of the fluid but assumed a discontinuous, cataract-like motion around the immersed body. In addition, Newton investigated the production of a (Cartesian) vortex through the rotation of a cylinder and thereby assumed shear stresses that transferred the motion for one coaxial layer of the fluid to the next. He also explained the propagation of sound through the elasticity of the air and thereby introduced the (normal) pressure between successive layers of the air.⁸

To sum up, Newton introduced two basic, long-lasting concepts of fluid mechanics: internal pressure (both longitudinal and transverse), and similarity. However, he had no general strategy for subjecting continuous media to the laws of his new mechanics. While his simplified models became popular, his concepts of internal pressure and similarity were long ignored. As we will see in a mo-

⁴ Euler 1755b.

⁵ Detailed presentations of these may be found in Truesdell's 1954 landmark work on Euler and fluid dynamics.

⁶ Grimberg, Pauls and Frisch 2008. Truesdell 1954: XXXVIII–XLI.

⁷ Cf. Truesdell 1954: IX–XIV; Rouse and Ince 1957: Chaps 2–9; Garbrecht 1987; Blay 1992; Eckert 2005: Chap. 1.

⁸ Cf. Smith 1998. Newton also discussed waves on water and the shape of a rotating fluid mass (figure of the Earth).

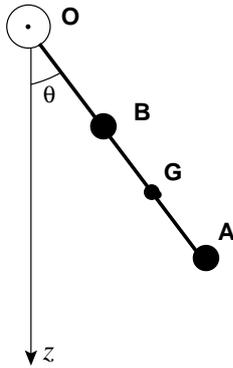


Figure 1 Compound pendulum

ment, much of the prehistory of Euler's equation has to do with the difficult reintroduction of internal pressure as a means to derive the motion of fluid elements. Although we are now accustomed to the idea that a continuum can be mentally decomposed into mutually pressing portions, this sort of abstraction long remained suspicious to the pioneers of Newtonian mechanics.

Daniel Bernoulli's Hydrodynamica

The Swiss physician and geometer Daniel Bernoulli was the first of these pioneers to develop a uniform dynamical method to solve a large class of problems of fluid motion. His reasoning was based on Leibniz's principle of live forces, and modeled after Huygens's influential treatment of the compound pendulum in his *Horologium oscillatorium* (1673).⁹

Consider a pendulum made of two point masses A and B rigidly connected to a massless rod that can oscillate around the suspension point O (fig. 1). Huygens required the equality of the "potential ascent" and the "actual descent," whose translation in modern terms reads:

$$\frac{m_A(v_A^2/2g) + m_B(v_B^2/2g)}{m_A + m_B} = z_G, \quad (2)$$

where m denotes a mass, v a velocity, g the acceleration of gravity, and z_G the descent of the gravity center of the two masses measured from the highest elevation of the pendulum during its oscillation. This equation, in which the modern reader recognizes the conservation of the sum of the kinetic and potential energies, leads to a first-order differential equation for the angle θ that the suspending rod makes with the vertical. The comparison of this equation with that of a simple pendulum then yields the expression $(a^2m_A + b^2m_B)/(am_A + bm_B)$ of the length of the equivalent simple pendulum (with $a = OA$ and $b = OB$).¹⁰

⁹ D. Bernoulli 1738; Huygens 1673.

¹⁰ Cf. Vilain 2000: 32–36.

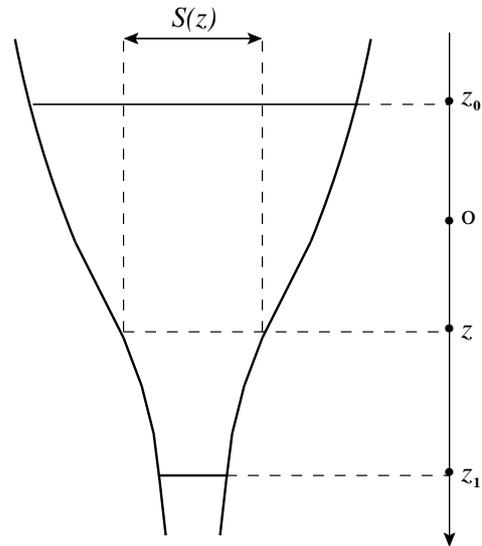


Figure 2 Parallel-slice flow in a vertical vessel.

As D. Bernoulli could not fail to observe, there is a close analogy between this problem and the hydraulic problem of efflux, as long as the fluid motion occurs by parallel slices. Under the latter hypothesis, the velocity of the fluid particles that belong to the same section of the fluid is normal to and uniform through the section. If, moreover, the fluid is incompressible and continuous (no cavitation), the velocity in one section of the vessel completely determines the velocity in all other sections. The problem is thus reduced to the fall of a connected system of weights with one degree of freedom only, just as is the case of a compound pendulum.

This analogy inspired D. Bernoulli's treatment of efflux. Consider, for instance, a vertical vessel with a section S depending on the downward vertical coordinate z (fig. 2). A mass of water falls through this vessel by parallel, horizontal slices. The continuity of the incompressible water implies that the product Sv is a constant through the fluid mass. The equality of the potential ascent and the actual descent implies that at every instant¹¹

$$\int_{z_0}^{z_1} \frac{v^2(z)}{2g} S(z) dz = \int_{z_0}^{z_1} z S(z) dz, \quad (3)$$

where z_0 and z_1 denote the (changing) coordinates of the two extreme sections of the fluid mass, the origin of the z -axis coincides with the position of the gravity center of this mass at the beginning of the fall, and the units are chosen so that the density of the fluid is *one*. As $v(z)$ is inversely proportional to the known function S of z , this equation yields a relation between z_0 and

¹¹ D. Bernoulli 1738: 31–35 gave a differential, geometric version of this relation.

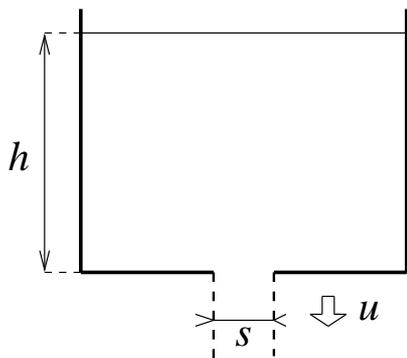


Figure 3 Idealized efflux through small opening (without *vena contracta*).

$v(z_0) = \dot{z}_0$, which can be integrated to give the motion of the highest fluid slice, and so forth. D. Bernoulli's investigation of efflux amounted to a repeated application of this procedure to vessels of various shapes.

The simplest sub-case of this problem is that of a broad container with a small opening of section s on its bottom (fig. 3). As the height h of the water varies very slowly, the escaping velocity quickly reaches a steady value u . As the fluid velocity within the vessel is negligible, the increase of the potential ascent in the time dt is simply given by the potential ascent $(u^2/2g)sudt$ of the fluid slice that escapes through the opening at the velocity u . This quantity must be equal to the actual descent $hsudt$. Therefore, the velocity u of efflux is the velocity $\sqrt{2gh}$ of free fall from the height h , in conformity with Torricelli's law.¹²

D. Bernoulli's most innovative application of this method concerned the pressure exerted by a moving fluid on the walls of its container, a topic of importance for the physician and physiologist he also was. Previous writers on hydraulics and hydrostatics had only considered the hydrostatic pressure due to gravity. In the case of a uniform gravity g , the pressure per unit area on a wall portion was known to depend only on the depth h of this portion below the free water surface. According to the law enunciated by Simon Stevin in 1605, it is given by the weight gh of a water column (of unit density) that has a unit normal section and the height h . In the case of a moving fluid, D. Bernoulli defined and derived the "hydraulic-static" wall pressure as follows.¹³

¹² D. Bernoulli 1738: 35. This reasoning assumes a parallel motion of the escaping fluid particle. Therefore, it only gives the velocity u beyond the contraction of the escaping fluid vein that occurs near the opening (Newton's *vena contracta*): cf. Lagrange 1788: 430–431; Smith 1998).

¹³ D. Bernoulli 1738: 258–260. Mention of physiological applications is found in D. Bernoulli to Shoepflin, 25 Aug 1734, in D. Bernoulli 2002: 89: "Hydraulic-statics will also be useful to understand animal economy with respect to the motion of fluids, their pressure on vessels, etc."

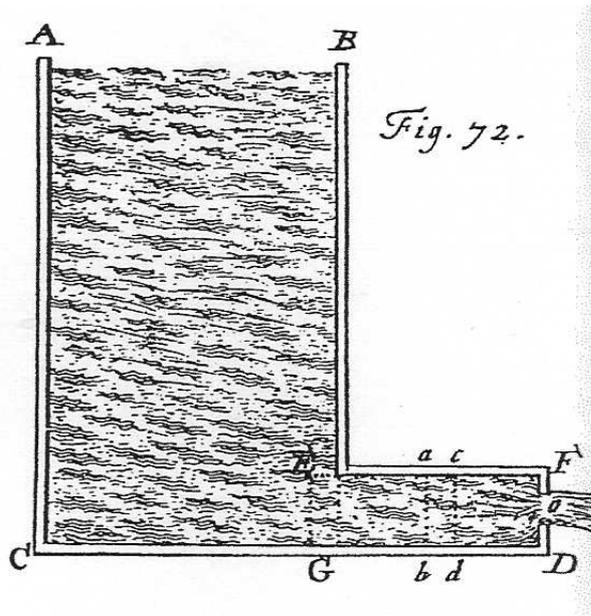


Figure 4 Daniel Bernoulli's figure accompanying his derivation of the velocity-dependence of pressure (1738: plate).

The section S of the vertical vessel ABCG of fig. 4 is supposed to be much larger than the section s of the appended tube EFDG, which is itself much larger than the section ε of the hole o . Consequently, the velocity u of the water escaping through o is $\sqrt{2gh}$. Owing to the conservation of the flux, the velocity v within the tube is $(\varepsilon/s)u$. D. Bernoulli goes on:¹⁴

If in truth there were no barrier FD, the final velocity of the water in the same tube would be [s/ε times greater]. Therefore, the water in the tube tends to a greater motion, but its pressing [*nisus*] is hindered by the applied barrier FD. By this pressing and resistance [*nisus et renisus*] the water is compressed [*compressitur*], which compression [*compressio*] is itself kept in by the walls of the tube, and thence these too sustain a similar pressure [*pressio*]. Thus it is plain that the pressure [*pressio*] on the walls is proportional to the acceleration... that would be taken on by the water if every obstacle to its motion should instantaneously vanish, so that it were ejected directly into the air.

Based on this intuition, D. Bernoulli imagined that the tube was suddenly broken at ab , and made the wall pressure P proportional to the acceleration dv/dt of the water at this instant. According to the principle of live forces, the actual descent of the water during the time dt must be equal to the potential ascent it acquires while passing from the large section S to the smaller section s , plus the increase of the potential ascent of the portion

¹⁴ D. Bernoulli 1738: 258–259, translated in Truesdell 1954: XXVII. The *compressio* in this citation perhaps prefigures the internal pressure later introduced by Johann Bernoulli.

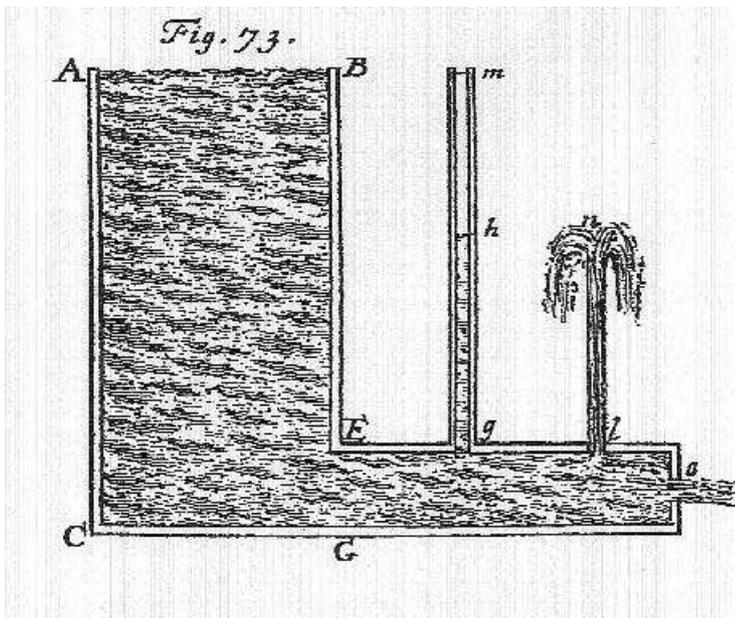


Figure 5 Effects of the velocity-dependence of pressure according to Daniel Bernoulli (1738: plate).

EabG of the fluid. This gives (the fluid density is one)

$$hsv dt = \frac{v^2}{2g} sv dt + bs d \left(\frac{v^2}{2g} \right), \quad (4)$$

where $b = Ea$. The resulting value of the acceleration dv/dt is $(gh - v^2/2)/b$. The wall pressure P must be proportional to this quantity, and it must be identical to the static pressure gh in the limiting case $v = 0$. It is therefore given by the equation

$$P = gh - \frac{1}{2}v^2, \quad (5)$$

which means that the pressure exerted by a moving fluid on the walls is lower than the static pressure, the difference being half the squared velocity (times the density). D. Bernoulli illustrated this effect in two manners (fig. 5): by connecting a narrow vertical tube to the horizontal tube EFDG, and by letting a vertical jet surge from a hole on this tube. Both reach a water level well below AB.

The modern reader may here recognize Bernoulli's law. In fact, D. Bernoulli did not quite write equation (5), because he chose the ratio s/ε rather than the velocity v as the relevant variable. Also, he only reasoned in terms of wall pressure, whereas modern physicists apply Bernoulli's law to the internal pressure of a fluid.

There were other limitations to D. Bernoulli's considerations, of which he was largely aware. He knew that in some cases part of the live force of the water went to eddying motion, and he even tried to estimate this loss in the case of a suddenly enlarged conduit. He was also aware of the imperfect fluidity of water, although he decided to ignore it in his reasoning. Most important, he

knew that the hypothesis of parallel slices only held for narrow vessels and for gradual variations of their section. But his method confined him to this case, since it is only for systems with one degree of freedom that the conservation of live forces suffices to determine the motion.¹⁵

To summarize, by means of the principle of live forces Daniel Bernoulli was able to solve many problems of quasi-one-dimensional flow and thereby related wall pressure to fluid velocity. This unification of hydrostatic and hydraulic considerations justified the title *Hydrodynamica* which he gave to the treatise he published in 1738 in Strasbourg. Besides the treatment of efflux, this work included all the typical questions of contemporary hydraulics except fluid resistance (which D. Bernoulli probably judged beyond the scope of his methods), a kinetic theory of gases, and considerations on Cartesian vortices. It is rightly regarded as a major turning point in the history of hydrodynamics, because of the uniformity and rigor of its dynamical method, the depth of physical insight, and the abundance of long-lasting results.¹⁶

Johann Bernoulli's *Hydraulica*

In 1742, Daniel's father Johann Bernoulli published his *Hydraulica*, with an antedate that made it seem anterior to his son's treatise. Although he had been the most ardent supporter of Leibniz's principle of live forces, he now regarded this principle as an indirect consequence of more fundamental laws of mechanics. His asserted aim was to base hydraulics on an incontrovertible, Newtonian expression of these laws. To this end he adapted a method he had invented in 1714 to solve the paradigmatic problem of the compound pendulum.

Consider again the pendulum of fig. 1. According to J. Bernoulli, the gravitational force $m_B g$ acting on B is equivalent to a force $(b/a)m_B g$ acting on A, because according to the law of levers two forces that have the same moment have the same effect. Similarly, the "accelerating force" $m_B b \ddot{\theta}$ of the mass B is equivalent to an accelerating force $(b/a)m_B b \ddot{\theta} = m_B (b/a)^2 a \ddot{\theta}$ at A. Consequently, the compound pendulum is equivalent to a simple pendulum with a mass $m_A + (b/a)^2 m_B$ located on A and subjected to the effective vertical force $m_A g + (b/a)m_B g$. It is also equivalent to a simple pendulum of length $(a^2 m_A + b^2 m_B)/(a m_A + b m_B)$ oscillating in the gravity g , in conformity with Huygens' result. In sum, Johann Bernoulli reached the equation of motion by applying Newton's second law to a fictitious system obtained by replacing the forces and the momentum variations at any point of the system with equivalent forces and momentum variations at one point of the system. This replace-

¹⁵ D. Bernoulli 1738: 12 (eddies), 124 (enlarged conduit); 13 (imperfect fluid).

¹⁶ On the *Hydrodynamica*, cf. Truesdell 1954: XXIII-XXXI; Calero 1996: 422-459; Mikhailov 2002.

ment, based on the laws of equilibrium of the system, is what J. Bernoulli called “translation” in the introduction to his *Hydraulica*.¹⁷

Now consider the canonical problem of water flowing by parallel slices through a vertical vessel of varying section (fig. 2). J. Bernoulli “translates” the weight $gSdz$ of the slice dz of the water to the location z_1 of the frontal section of the fluid. This gives the effective weight S_1gdz , because according to a well-known law of hydrostatics a pressure applied at any point of the surface of a confined fluid is uniformly transmitted to any other part of the surface of the fluid. Similarly, J. Bernoulli translates the “accelerating force” (momentum variation) $(dv/dt)Sdz$ of the slice dz to the frontal section of the fluid, with the result $(dv/dt)S_1dz$. He then obtains the equation of motion by equating the total translated weight to the total translated accelerating force:

$$S_1 \int_{z_0}^{z_1} g dz = S_1 \int_{z_0}^{z_1} \frac{dv}{dt} dz. \quad (6)$$

For J. Bernoulli the crucial point was the determination of the acceleration dv/dt . Previous authors, he contended, had failed to derive correct equations of motion from the general laws of mechanics because they were only aware of one contribution to the acceleration of the fluid slices: that which corresponds to the instantaneous change of velocity at a given height z , or $\partial v/\partial t$ in modern terms. They ignored the acceleration due to the broadening or to the narrowing of the section of the vessel, which J. Bernoulli called *gorges* (gorge). In modern terms, he identified the convective component $v(\partial v/\partial z)$ of the acceleration. Note that his use of partial derivatives was only implicit: thanks to the relation $v = (S_0/S)v_0$, he could split v into a time dependent factor v_0 and a z -dependent factor S_0/S and thus express the total acceleration as $(S_0/S)(dv_0/dt) - (v_0^2 S_0^2/S^3)(dS/dz)$.¹⁸

Thanks to the *gorges*, J. Bernoulli successfully applied equation (6) to various cases of efflux and retrieved his son’s results.¹⁹ He also offered a novel approach to the pressure of a moving fluid on the side of its container. This pressure, he asserted, was nothing but the pressure, or *vis immaterialis* that contiguous fluid parts exerted

on one another, just as two solids in contact act on each other.²⁰

The force that acts on the side of the channel through which the liquid flows... is nothing but the force that originates in the force of compression through which contiguous parts of the fluid act on one another.

Accordingly, J. Bernoulli divided the flowing mass of water into two parts separated by the section $z = \zeta$. Following the general idea of “translation,” the pressure that the upper part exerts on the lower part is

$$P(\zeta) = \int_{z_0}^{\zeta} (g - dv/dt) dz. \quad (7)$$

More explicitly, this is

$$\begin{aligned} P(\zeta) &= \int_{z_0}^{\zeta} g dz - \int_{z_0}^{\zeta} v \frac{\partial v}{\partial z} dz - \int_{z_0}^{\zeta} \frac{\partial v}{\partial t} dz \\ &= g(\zeta - z_0) - \frac{1}{2}v^2(\zeta) + \frac{1}{2}v^2(z_0) - \frac{\partial}{\partial t} \int_{z_0}^{\zeta} v dz. \end{aligned} \quad (8)$$

In a widely different notation, J. Bernoulli thus obtained a generalization of his son’s law to non-stationary parallel-slice flow.²¹

J. Bernoulli interpreted the relevant pressure as an *internal* pressure analogous to the tension of a thread or the mutual action of contiguous solids in connected systems. Yet he did not rely on this new concept of pressure to establish the equation of motion (6). He only introduced this concept as a short-cut to the velocity-dependence of wall-pressure.²²

To summarize, Johann Bernoulli’s *Hydraulica* departed from his son’s *Hydrodynamica* through a more direct reliance on Newton’s laws. This approach required the new concept of convective derivative. It permitted a generalization of Bernoulli’s law to the pressure in a non-steady flow. J. Bernoulli had a concept of internal pressure, although he did not use it in his derivation of the equation of fluid motion. Like his son’s, his dynamical method was essentially confined to systems with one degree of freedom only, so that he could only treat flow by parallel slices.

¹⁷ J. Bernoulli 1714; 1742: 395. In modern terms, J. Bernoulli’s procedure amounts to equating the sum of moments of the applied forces to the sum of moments of the accelerating forces (which is the time derivative of the total angular momentum). Cf. Vilain 2000: 448–450.

¹⁸ J. Bernoulli 1742: 432–437. He misleadingly called the two parts of the acceleration the “hydraulic” and the “hydrostatic” components. Truesdell (1954: XXXIII) translates *gorges* as “eddy” (it does have this meaning in classical latin), because in the case of sudden (but small) decrease of section J. Bernoulli imagined a tiny eddy at the corners of the gorge. In his treatise on the equilibrium and motion of fluids (1744: 157), d’Alembert interpreted J. Bernoulli’s expression of the acceleration in terms of two partial differentials.

¹⁹ D’Alembert later explained this agreement: see below, pp. 7–8.

²⁰ J. Bernoulli 1742: 442.

²¹ J. Bernoulli 1742: 444. His notation for the internal pressure was π . In the first section of his *Hydraulica*, which he communicated to Euler in 1739, he only treated the steady flow in a suddenly enlarged tube. In his enthusiastic reply (5 May 1739, in Euler 1998: 287–295), Euler treated the accelerated efflux from a vase of arbitrary shape with the same method of “translation,” not with the later method of balancing gravity with internal pressure gradient, contrary to Truesdell’s claim (1954: XXXIII). J. Bernoulli subsequently wrote his second part, where he added the determination of the internal pressure to Euler’s treatment.

²² For a different view, cf. Truesdell 1954: XXXIII; Calero 1996: 460–474.

III. D'ALEMBERT'S FLUID DYNAMICS

The principle of dynamics

In 1743, the French geometer and philosopher Jean le Rond d'Alembert published his influential *Traité de dynamique*, which subsumed the dynamics of connected systems under a few general principles. The first illustration he gave of his approach was Huygens's compound pendulum. As we saw, Johann Bernoulli's solution to this problem leads to the equation of motion

$$m_A g \sin \theta + (b/a)m_B g \sin \theta = m_A a \ddot{\theta} + (b/a)m_B b \ddot{\theta}, \quad (9)$$

which may be rewritten as

$$a(m_A g \sin \theta - m_A a \ddot{\theta}) + b(m_B g \sin \theta - m_B b \ddot{\theta}) = 0. \quad (10)$$

The latter is the condition of equilibrium of the pendulum under the action of the forces $m_A \mathbf{g} - m_A \boldsymbol{\gamma}_A$ and $m_B \mathbf{g} - m_B \boldsymbol{\gamma}_B$ acting respectively on A and B. In d'Alembert's terminology, the products $m_A \mathbf{g}$ and $m_B \mathbf{g}$ are the motions impressed (per unit time) on the bodies A and B under the sole effect of gravitation (without any constraint). The products $m_A \boldsymbol{\gamma}_A$ and $m_B \boldsymbol{\gamma}_B$ are the actual changes of their (quantity of) motion (per unit time). The differences $m_A \mathbf{g} - m_A \boldsymbol{\gamma}_A$ and $m_B \mathbf{g} - m_B \boldsymbol{\gamma}_B$ are the parts of the impressed motions that are destroyed by the rigid connection of the two masses through the freely rotating rod. Accordingly, d'Alembert saw in equation (10) a consequence of a general dynamic principle following which the motions destroyed by the connections should be in equilibrium.²³

D'Alembert based his dynamics on three laws, which he regarded as necessary consequences of the principle of sufficient reason. The first law is that of inertia, according to which a freely moving body moves with a constant velocity in a constant direction. The second law stipulates the vector superposition of motions impressed on a given body. According to the third law, two (ideally rigid) bodies come to rest after a head-on collision if and only if their velocities are inversely proportional to their masses. From these three laws and further recourse to the principle of sufficient reason, d'Alembert believed he could derive a complete system of dynamics without recourse to the older, obscure concept of force as cause of motion. He defined force as the motion impressed on a body, that is, the motion that a body would take if this force were acting alone without any impediment. Then the third law implies that two contiguous bodies subjected to opposite forces are in equilibrium. More generally, d'Alembert regarded statics as a particular case of

dynamics in which the various motions impressed on the parts of the system mutually cancel each other.²⁴

Based on this conception, d'Alembert derived the principle of virtual velocities according to which a connected system subjected to various forces remains in equilibrium if the work of these forces vanishes for any infinitesimal motion of the system that is compatible with the connections.²⁵ As for the principle of dynamics, he regarded it as a self-evident consequence of his dynamic concept of equilibrium. In general, the effect of the connections in a connected system is to destroy part of the motion that is impressed on its components by means of external agencies. The rules of this destruction should be the same whether the destruction is total or partial. Hence, equilibrium should hold for that part of the impressed motions that is destroyed through the constraints. This is d'Alembert's principle of dynamics. Stripped of d'Alembert's philosophy of motion, this principle stipulates that a connected system in motion should be at any time in equilibrium with respect to the fictitious forces $\mathbf{f} - m\boldsymbol{\gamma}$, where \mathbf{f} denotes the force applied on the mass point m of the system, and $\boldsymbol{\gamma}$ the acceleration of this mass point.

Efflux revisited

At the end of his treatise on dynamics, d'Alembert considered the hydraulic problem of efflux through the vessel of fig. 2. His first task was to determine the condition of equilibrium of the fluid when subjected to an altitude-dependent gravity $g(z)$. For this purpose he considered an intermediate slice of the fluid, and required the pressure from the fluid above this slice to be equal and opposite to the pressure from the fluid below this slice. According to a slight generalization of Stevin's hydrostatic law, these two pressures are given by the integral of the variable gravity $g(z)$ over the relevant range of elevation. Hence the equilibrium condition reads:²⁶

$$S(\zeta) \int_{z_0}^{\zeta} g(z) dz = -S(\zeta) \int_{\zeta}^{z_1} g(z) dz, \quad (11)$$

or

$$\int_{z_0}^{z_1} g(z) dz = 0. \quad (12)$$

According to d'Alembert's principle, the motion of the fluid under a constant gravity g must be such that the fluid be in equilibrium under the fictitious gravity $g(z) =$

²³ D'Alembert 1743: 69–70. Cf. Vilain 2000: 456–459. D'Alembert reproduced and criticized Johann Bernoulli's derivation on p. 71. On Jacob Bernoulli's anticipation of d'Alembert's principle, cf. Lagrange 1788: 176–177, 179–180; Dugas 1950: 233–234; Vilain 2000: 444–448.

²⁴ D'Alembert 1743: xiv–xv, 3. Cf. Hanksins 1968; Fraser 1985.

²⁵ The principle of virtual velocities was first stated generally by Johann Bernoulli and thus named by Lagrange (1788: 8–11). Cf. Dugas 1950: 221–223, 320. The term work is of course anachronistic.

²⁶ D'Alembert 1743: 183–186.

$g - dv/dt$, where dv/dt is the acceleration of the fluid slice at the elevation z . Hence comes the equation of motion

$$\int_{z_0}^{z_1} \left(g - \frac{dv}{dt} \right) dz = 0, \quad (13)$$

which is the same as Johann Bernoulli's equation (6). In addition, d'Alembert proved that this equation, together with the constancy of the product Sv , implied the conservation of live forces in Daniel Bernoulli's form (equation (3)).

In his subsequent treatise of 1744 on the equilibrium and motion of fluids, d'Alembert provided a similar treatment of efflux including his earlier derivations of the equation of motion and the conservation of live forces, with a slight variant: he now derived the equilibrium condition (13) by setting the pressure acting on the bottom slice of the fluid to zero.²⁷ Presumably, he did not want to base the equations of equilibrium and motion on the concept of internal pressure, in conformity with his general avoidance of internal contact forces in his dynamics. His statement of the general conditions of equilibrium of a fluid, as found at the beginning of his treatise, only required the concept of wall-pressure. Yet, in a later section of his treatise d'Alembert introduced "the pressure at a given height"

$$P(\zeta) = \int_{z_0}^{\zeta} (g - dv/dt) dz, \quad (14)$$

just as Johann Bernoulli had done, and for the same purpose of deriving the velocity dependence of wall-pressure.²⁸

In the rest of his treatise, d'Alembert solved problems similar to those of Daniel Bernoulli's *Hydrodynamica*, with nearly identical results. The only important difference concerned cases involving the sudden impact of two layers of fluids. Whereas Daniel Bernoulli still applied the conservation of live forces in such cases (save for possible dissipation into turbulent motion), d'Alembert's principle of dynamics there implied a destruction of live force. Daniel Bernoulli disagreed with these and a few other changes. In a contemporary letter to Euler he expressed his exasperation over d'Alembert's treatise:²⁹

I have seen with astonishment that apart from a few little things there is nothing to be seen in his hydrodynamics but an impertinent conceit. His criticisms are puerile indeed, and show not only that he is no remarkable man, but also that he never will be.³⁰

²⁷ Ibid.: 19–20.

²⁸ Ibid.: 139.

²⁹ D. Bernoulli to Euler, 7 Jul 1745, quoted in Truesdell 1954: XXXVIIIn.

³⁰ This is but an instance of the many cutting remarks exchanged between eighteenth-century geometers; further examples are not needed here.

The cause of winds

In this judgment, Daniel Bernoulli overlooked that d'Alembert's hydrodynamics, being based on a general dynamics of connected systems, lent itself to generalizations beyond parallel-slice flow. D'Alembert offered striking illustrations of the power of his approach in a prize-winning memoir published in 1747 on the cause of winds.³¹ As thermal effects were beyond the grasp of contemporary mathematical physics, he focused on a cause that is now known to be negligible: the tidal force exerted by the luminaries (the Moon and the Sun). For simplicity, he confined his analysis to the case of a constant-density layer of air covering a spherical globe with uniform thickness. He further assumed that fluid particles originally on the same vertical line remained so in the course of time and that the vertical acceleration of these particles was negligible (owing to the thinness of the air layer), and he neglected second-order quantities with respect to the fluid velocity and to the elevation of the free surface. His strategy was to apply his principle of dynamics to the motion induced by the tidal force \mathbf{f} and the terrestrial gravity \mathbf{g} , both of which depend on the location on the surface of the Earth.³²

Calling γ the absolute acceleration of the fluid particles, the principle requires that the fluid layer should be in equilibrium under the force $\mathbf{f} + \mathbf{g} + \gamma$ (the density of the air is *one* in the chosen units). From earlier theories of the shape of the Earth (regarded as a rotating liquid spheroid) d'Alembert borrowed the equilibrium condition that the net force should be perpendicular to the free surface of the fluid. He also required that the volume of vertical cylinders of fluid should not be altered by their motion, in conformity with his constant-density model. As the modern reader would expect, from these two conditions d'Alembert derived some sort of momentum equation, and some sort of incompressibility equation. He did so in a rather opaque manner. Some features, such as the lack of specific notation for partial differentials or the abundant recourse to geometrical reasoning, disconcert

³¹ As a member of the committees judging the Berlin Academy's prizes on winds and on fluid resistance (he could not compete as a resident member), Euler studied d'Alembert's submitted memoirs of 1747 and 1749. The subject set for the first prize, probably written by Euler, was "to determine the order & the law wind should follow, if the Earth were surrounded on all sides by the Ocean; so that one could at all times predict the speed & direction of the wind in all places." The question is here formulated in terms of what we now call Eulerian coordinates ("all places"), cf. Grimberg 1998:195.

³² D'Alembert 1747. D'Alembert treated the rotation of the Earth, the attraction by the Sun and the Moon as small perturbing causes whose effects on the shape of the fluid surface simply added (ibid.: xvii, 47). Consequently, he overlooked the Coriolis force in his analysis of the tidal effects (ibid.: 65, he writes he will be doing as if it were the luminary that rotates around the Earth).

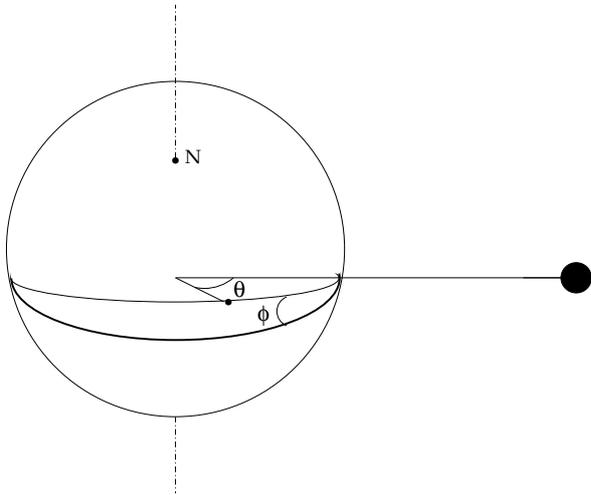


Figure 6 Spherical coordinates for d’Alembert’s atmospheric tides. The fat line represents the visible part of the equator, over which the luminary is orbiting. N is the North pole.

modern readers only.³³ Others were problematic to his contemporaries: he often omitted steps and introduced special assumptions without warning. Also, he directly treated the utterly difficult problem of fluid motion on a spherical surface without preparing the reader with simpler problems.

Suppose, with d’Alembert, that the tide-inducing luminary orbits above the equator (with respect to the Earth).³⁴ Using the modern terminology for spherical coordinates, call θ the colatitude of a given point of the terrestrial sphere with respect to an axis pointing toward the orbiting luminary, ϕ the longitude measured from the meridian above which the luminary is orbiting (this is *not* the geographical longitude), η the elevation of the free surface of the fluid layer over its equilibrium position, v_θ and v_ϕ the θ - and ϕ -components of the fluid velocity with respect to the Earth, h the depth of the fluid in its undisturbed state, and R the radius of the Earth (see fig. 6).

D’Alembert first considered the simpler case when ϕ is negligibly small, for which he expected the component v_ϕ also to be negligible. To first order in η and v , the conservation of the volume of a vertical column of fluid yields

$$\frac{1}{h} \dot{\eta} + \frac{1}{R} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta}{R \tan \theta} = 0, \quad (15)$$

which means that an increase of the height of the column is compensated by a narrowing of its basis (the dot denotes the time derivative at a fixed point of the Earth

surface). Since the tidal force \mathbf{f} is much smaller than the gravity \mathbf{g} , the vector sum $\mathbf{f} + \mathbf{g} - \boldsymbol{\gamma}$ makes an angle $(f_\theta - \gamma_\theta)/g$ with the vertical. To first order in η , the inclination of the fluid surface over the horizontal is $(\partial\eta/\partial\theta)/R$. Therefore, the condition that $\mathbf{f} + \mathbf{g} - \boldsymbol{\gamma}$ should be perpendicular to the surface of the fluid is approximately identical to³⁵

$$\gamma_\theta = f_\theta - \frac{g}{R} \frac{\partial \eta}{\partial \theta}. \quad (16)$$

As d’Alembert noted, this equation of motion can also be obtained by equating the horizontal acceleration of a fluid slice to the sum of the tidal component f_θ and of the difference between the pressures on both sides of this slice. Indeed, the neglect of the vertical acceleration implies that at a given height, the internal pressure of the fluid varies as the product $g\eta$. Hence d’Alembert was aware of two routes to the equation of motion, through his dynamic principle, or through an application of the momentum law to a fluid element subjected to the pressure of contiguous elements. In some sections he favored the first route, in others the second.³⁶

In his expression of the time variations $\dot{\eta}$ and \dot{v}_θ , d’Alembert considered only the forced motion of the fluid for which the velocity field and the free surface of the fluid rotate together with the tide-inducing luminary at the angular velocity $-\omega$. Then the values of η and v_θ at the colatitude θ and at the time $t + dt$ are equal to their values at the colatitude $\theta + \omega dt$ and at the time t . This gives

$$\dot{v}_\theta = \omega \frac{\partial v_\theta}{\partial \theta}, \quad \dot{\eta} = \omega \frac{\partial \eta}{\partial \theta}. \quad (17)$$

D’Alembert equated the relative acceleration \dot{v}_θ with the acceleration γ_θ , for he neglected the second-order convective terms, and judged the absolute rotation of the Earth irrelevant (he was aware of the centripetal acceleration, but treated the resulting permanent deformation of the fluid surface separately; and he overlooked the Coriolis acceleration). With these substitutions, his equations (15) and (16) become ordinary differential equations with respect to the variable θ .

³³ D’Alembert used a purely geometrical method to study the free oscillations of an ellipsoidal disturbance of the air layer.

³⁴ The sun and the moon actually do not, but the *variable* part of their action is proportional to that of such a luminary.

³⁵ D’Alembert 1747: 88–89 (formulas A and B). The correspondence with d’Alembert’s notation is given by: $\theta \mapsto u$, $v_\theta \mapsto q$, $\partial\eta/\partial\theta \mapsto -v$, $R/h\omega \mapsto \varepsilon$, $R/gK \mapsto 3S/4pd^3$ (with $f = -K \sin 2\theta$).

³⁶ D’Alembert 1747: 88–89. He represented the internal pressure by the weight of a vertical column of fluid. In his discussion of the condition of equilibrium (1747: 15–16), he introduced the balance of the horizontal component of the external force acting on a fluid element and the difference of weight of the two adjacent columns as “another very easy method” for determining the equilibrium. In the case of tidal motion with $\phi \approx 0$, he directly applied this condition of equilibrium to the “destroyed motion” $\mathbf{f} + \mathbf{g} - \boldsymbol{\gamma}$. In the general case (ibid.: 112–113), he used the perpendicularity of $\mathbf{f} + \mathbf{g} - \boldsymbol{\gamma}$ to the free surface of the fluid.

D'Alembert eliminated η from these two equations, and integrated the resulting differential equation for Newton's value $-K \sin 2\theta$ of the tide-inducing force f_θ . In particular, he showed that the phase of the tides (concordance or opposition) depended on whether the rotation period $2\pi/\omega$ of the luminary was smaller or larger than the quantity $2\pi R/\sqrt{gh}$, which he had earlier shown to be identical with the period of the free oscillations of the fluid layer.³⁷

In another section of his memoir, d'Alembert extended his equations to the case when the angle ϕ is no longer negligible. Again, he had the velocity field and the free surface of the fluid rotate together with the luminary at the angular velocity $-\omega$. Calling $\mathbf{R}_{\omega dt}$ the operator for the rotation of angle ωdt around the axis joining the center of the Earth and the luminary and $\mathbf{v}(\mathbf{P}, t)$ the velocity vector at point P and at time t , we have

$$\mathbf{v}(\mathbf{P}, t + dt) = \mathbf{R}_{\omega dt} \mathbf{v}(\mathbf{R}_{\omega dt} \mathbf{P}, t). \quad (18)$$

Expressing this relation in spherical coordinates, d'Alembert obtained

$$\dot{v}_\theta = \omega \left(\frac{\partial v_\theta}{\partial \theta} \cos \phi - \frac{\partial v_\theta}{\partial \phi} \frac{\sin \phi}{\tan \theta} - v_\phi \sin \phi \sin \theta \right), \quad (19)$$

$$\dot{v}_\phi = \omega \left(\frac{\partial v_\phi}{\partial \theta} \cos \phi - \frac{\partial v_\phi}{\partial \phi} \frac{\sin \phi}{\tan \theta} + v_\theta \sin \phi \sin \theta \right). \quad (20)$$

For the same reasons as before, d'Alembert identified these derivatives with the accelerations γ_θ and γ_ϕ . He then applied his dynamic principle to get

$$\gamma_\theta = f_\theta - \frac{g}{R} \frac{\partial \eta}{\partial \theta}, \quad (21)$$

$$\gamma_\phi = -\frac{g}{R \sin \theta} \frac{\partial \eta}{\partial \phi}. \quad (22)$$

Lastly, he obtained the continuity condition:

$$\begin{aligned} \dot{\eta} &= \omega \left(\frac{\partial \eta}{\partial \theta} \cos \phi - \frac{\partial \eta}{\partial \phi} \frac{\sin \phi}{\tan \theta} \right) \\ &= -\left(\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta}{\tan \theta} + \frac{1}{\sin \theta} \frac{\partial v_\phi}{\partial \phi} \right), \end{aligned} \quad (23)$$

in which the modern reader recognizes the expression of a divergence in spherical coordinates.³⁸

D'Alembert judged the resolution of this system to be beyond his capability. The purpose of this section of his memoir was to illustrate the power and generality of his method for deriving hydrodynamic equations. For the first time, he gave the complete equations of motion of an incompressible fluid in a genuinely two-dimensional case. Thus emerged the velocity field and partial derivatives with respect to two independent spatial coordinates. Although Alexis Fontaine and Euler had earlier developed the needed calculus of differential forms, d'Alembert was first to apply it to the dynamics of continuous media. His notation of course differed from the modern one: where we now write $\partial f/\partial x$, Fontaine wrote df/dx , and d'Alembert often wrote A , with $df = A dx + B dy + \dots$

The resistance of fluids

In 1749 d'Alembert submitted a Latin manuscript on the resistance of fluids for another Berlin prize, and failed. The Academy judged that none of the competitors had reached the point of comparing his theoretical results with experiments. D'Alembert did not deny the importance of this comparison for the improvement of ship design. But he judged that the relevant equations could not be solved in a near future, and that his memoir deserved consideration for its methodological innovations. In 1752, he published an augmented translation of this memoir as a book.³⁹

Compared with the earlier treatise on the equilibrium and motion of fluids, the first important difference was a new formulation of the laws of hydrostatics. In 1744, d'Alembert started with the uniform and isotropic transmissibility of pressure by any fluid (from one part of its surface to another). He then derived the standard laws of this science, such as the horizontality of the free surface and the depth-dependence of wall-pressure, by qualitative or geometrical reasoning. In contrast, in his new memoir he relied on a mathematical principle borrowed from Alexis-Claude Clairaut's memoir of 1743 on the shape of the Earth. According to this principle, a fluid mass subjected to a force density \mathbf{f} is in equilibrium if and only if the integral $\int \mathbf{f} \cdot d\mathbf{l}$ vanishes over any closed loop within the fluid and over any path whose ends belong to the free surface of the fluid.⁴⁰

D'Alembert regarded this principle as a mathematical expression of his earlier principle of the uniform transmissibility of pressure. If the fluid is globally in equilibrium,

³⁷ The elimination of η leads to the easily integrable equation $(gh - R^2\omega^2)dv_\theta + gh d(\sin \theta)/\sin \theta - R^2\omega K \sin \theta d(\sin \theta) = 0$.

³⁸ D'Alembert 1747: 111–114 (equations E, F, G, H, I). To complete the correspondence given in note (37), take $\phi \mapsto A$, $v_\theta \mapsto \eta$, $\gamma_\theta \mapsto \pi$, $\gamma_\phi \mapsto \varphi$, $g/R \mapsto p$, $\partial\eta/\partial\theta \mapsto -\rho$, $\partial\eta/\partial\phi \mapsto -\sigma$, $\partial v_\theta/\partial\theta \mapsto r$, $\partial v_\theta/\partial\phi \mapsto \lambda$, $\partial v_\phi/\partial\theta \mapsto \gamma$, $\partial v_\phi/\partial\phi \mapsto \beta$. D'Alembert has the ratio of two sines instead of the product in the last term of equations (19)–(20). An easy, modern way to obtain these equations is to rewrite (18) as $\dot{\mathbf{v}} = [(\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla] \mathbf{v} + \boldsymbol{\omega} \times \mathbf{v}$, with $\mathbf{v} = (\mathbf{0}, v_\theta, v_\phi)$, $\mathbf{r} = (R, \mathbf{0}, \mathbf{0})$, $\boldsymbol{\omega} = \omega(\sin \theta \sin \phi, \cos \theta \sin \phi, \cos \phi)$, and $\nabla = (\partial_r, \partial_\theta/R, \partial_\phi/(R \sin \theta))$ in the local basis.

³⁹ D'Alembert 1752: xxxviii. For an insightful study of d'Alembert's work on fluid resistance, cf. Grimberg 1998 (which also contains a transcript of the Latin manuscript submitted for the Berlin prize). See also Calero 1996: Chap. 8.

⁴⁰ D'Alembert 1752: 14–17. On the early history of theories of the figure of the Earth, cf. Todhunter 1873. On Clairaut, cf. Passeron 1995. On Clairaut's principle and Newton's and MacLaurin's partial anticipations, cf. Truesdell 1954: XIV–XXII.

he reasoned, it must also be in equilibrium within any narrow canal of section ε belonging to the fluid mass. For a canal beginning and ending on the free surface of the fluid, the pressure exerted by the fluid on each of the extremities of the canal must vanish. According to the principle of uniform transmissibility of pressure, the force \mathbf{f} acting on the fluid within the length $d\mathbf{l}$ of the canal exerts a pressure $\varepsilon \mathbf{f} \cdot d\mathbf{l}$ that is transmitted to both ends of the canal (with opposite signs). As the sum of these pressures must vanish, so does the integral $\int \mathbf{f} \cdot d\mathbf{l}$. This reasoning and a similar one for closed canals establish d'Alembert's new principle of equilibrium.⁴¹

Applying this principle to an infinitesimal loop, d'Alembert obtained (the Cartesian-coordinate form of) the differential condition

$$\nabla \times \mathbf{f} = \mathbf{0}, \quad (24)$$

as Clairaut had already done. Combining it with his principle of dynamics, and confining himself to the steady motion ($\partial \mathbf{v} / \partial t = \mathbf{0}$, so that $\boldsymbol{\gamma} = (\mathbf{v} \cdot \nabla) \mathbf{v}$) of an incompressible fluid, he obtained the two-dimensional, Cartesian-coordinate version of

$$\nabla \times [(\mathbf{v} \cdot \nabla) \mathbf{v}] = \mathbf{0}, \quad (25)$$

which means that the fluid must formally be in equilibrium with respect to the convective acceleration. D'Alembert then showed that this condition was met whenever $\nabla \times \mathbf{v} = \mathbf{0}$. Confusing a sufficient condition with a necessary one, he concluded that the latter property of the flow held generally.⁴²

This property nonetheless holds in the special case of motion investigated by d'Alembert, that is, the stationary flow of an incompressible fluid around a solid body when the flow is uniform far away from the body (fig. 7). In this limited case, d'Alembert gave a correct proof of which a modernized version follows.⁴³

Consider two neighboring lines of flow beginning in the uniform region of the flow and ending in any other part of the flow, and connect the extremities through a small segment. According to d'Alembert's principle together with the principle of equilibrium the integral $\oint (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot d\mathbf{r}$ vanishes over this loop. Using the identity

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = \nabla \left(\frac{1}{2} \mathbf{v}^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}), \quad (26)$$

⁴¹ As is obvious to the modern reader, this principle is equivalent to the existence of a single-valued function (P) of which \mathbf{f} is the gradient and which has a constant value on the free surface of the fluid. The canal equilibrium results from the principle of solidification, the history of which is discussed in Casey 1992.

⁴² D'Alembert 1752: art. 78. The modern hydrodynamicist recognizes in equation (25) a particular case of the vorticity equation. The condition $\nabla \times \mathbf{v} = \mathbf{0}$ is that of irrotational flow.

⁴³ For a more literal rendering of d'Alembert's proof, cf. Grimberg 1998: 43–48.

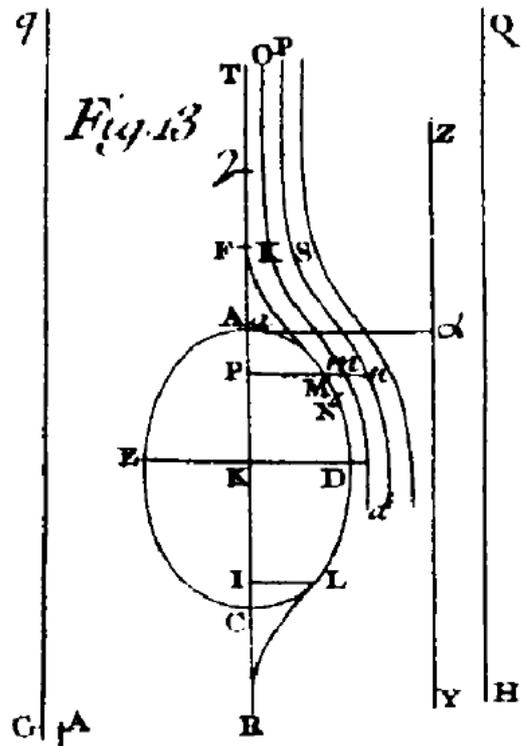


Figure 7 Flow around a solid body according to d'Alembert (1752: plate 13).

this implies that the integral $\oint (\nabla \times \mathbf{v}) \cdot (\mathbf{v} \times d\mathbf{r})$ also vanishes. The only part of the loop that contributes to this integral is that corresponding to the little segment joining the end points of the two lines of flow. Since the orientation of this segment is arbitrary, $\nabla \times \mathbf{v}$ must vanish.

D'Alembert thus derived the condition

$$\nabla \times \mathbf{v} = \mathbf{0} \quad (27)$$

from his dynamical principle. In addition he obtained the (incompressibility) condition

$$\nabla \cdot \mathbf{v} = 0 \quad (28)$$

by considering the deformation of a small parallelepiped of fluid during an infinitesimal time interval. More exactly, he obtained the special expressions of these two conditions in the two-dimensional case and in the axially-symmetric case. In the latter case, he wrote the incompressibility condition as

$$\frac{dq}{dx} + \frac{dp}{dz} = \frac{p}{z}, \quad (29)$$

where z and x are the radial and axial coordinates and p and q the corresponding components of the velocity. D'Alembert's 1749 derivation (repeated in his 1752 book)

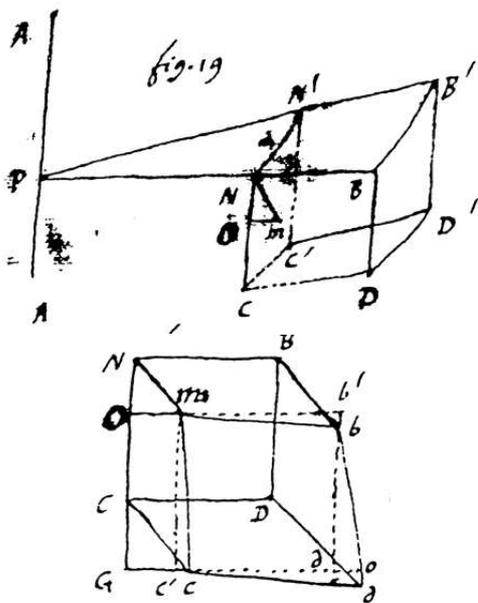


Figure 8 D'Alembert's drawing for a first proof of the incompressibility condition. He takes an infinitesimal prismatic volume $NBDCN'B'D'$ (upper figure). The faces $NBDC$ and $N'B'D'C'$ are rectangles in planes passing through the axis of symmetry AP ; After an infinitesimal time dt the points $NBDC$ have moved to $nbdc$ (lower figure). Expressing the conservation of volume and neglecting higher-order infinitesimals, he obtains equation (29). From the 1749 manuscript in the Berlin-Brandenburgische Akademie der Wissenschaften; courtesy Wolfgang Knobloch and Gérard Grimberg.

is illustrated by a geometrical construction (fig. 8).⁴⁴

In order to solve the system (27)-(28) in the two-dimensional case, d'Alembert noted that the two conditions meant that the forms $u dx + v dy$ and $v dx - u dy$ were exact differentials (u and v denote the velocity components along the orthogonal axes Ox and Oy). This property holds, he ingeniously noted, if and only if $(u - iv)(dx + idy)$ is an exact differential. This means that u and $-v$ are the real and imaginary parts of a (holomorphic) function of the complex variable $x + iy$. They must also be such that the velocity be uniform at infinity and tangent to the body along its surface. D'Alembert struggled to meet these boundary conditions through power-series developments, to little avail.⁴⁵

⁴⁴ It thus would seem appropriate to use "d'Alembert's condition" when referring to the condition of incompressibility, written as a partial differential equation.

⁴⁵ D'Alembert 1752: 60–62. D'Alembert here discovered the Cauchy–Riemann condition for u and $-v$ to be the real and imaginary components of an analytic function in the complex plane, as well as a powerful method to solve Laplace's equation $\Delta u = 0$ in two dimensions. In 1761: 139, d'Alembert introduced the complex potential $\varphi + i\psi$ such that $(u - iv)(dx + idy) = d(\varphi + i\psi)$. The real part φ of this potential is the velocity potential intro-

The ultimate goal of this calculation was to determine the force exerted by the fluid on the solid, which is the same as the resistance offered by the fluid to the motion of a body with a velocity opposite to that of the asymptotic flow.⁴⁶ D'Alembert expressed this force as the integral of the fluid's pressure over the whole surface of the body. The pressure is itself given by the line integral of $-dv/dt$ from infinity to the wall, in conformity with d'Alembert's earlier derivation of Bernoulli's law. This law still holds in the present case, because $-dv/dt = -(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla(v^2/2)$. Hence the resistance could be determined, if only the flow around the body was known.⁴⁷

D'Alembert was not able to solve his equations and to truly answer the resistance question. Yet, he had achieved much on the way: through his dynamical principle and his equilibrium principle, he had obtained hydrodynamic equations for the steady flow of an incompressible axisymmetrical flow that we may retrospectively identify as the incompressibility condition, the condition of irrotational flow, and Bernoulli's law. The modern reader may wonder why he did not try to write general equations of fluid motion in Cartesian-coordinate form. The answer is plain: he was following an older tradition of mathematical physics according to which general principles, rather than general equations, were applied to specific problems.

D'Alembert obtained his basic equations without recourse to the concept of pressure. Yet he had a concept of internal pressure, which he used to derive Bernoulli's law. Curiously, he did not pursue the other approach sketched in his theory of winds, that is, the application of Newton's second law to a fluid element subjected to a pressure gradient. Plausibly, he favored a derivation that was based on his own principle of dynamics and thus avoided the kind of internal forces he judged obscure.

It was certainly well known to d'Alembert that his equilibrium principle was nothing but the condition of uniform integrability (potentiality) for the force density \mathbf{f} . If one then introduces the integral, say P , one ob-

duced by Euler in 1752; its imaginary part ψ is the so-called stream function, which is a constant on any line of current, as d'Alembert noted.

⁴⁶ D'Alembert gave a proof of this equivalence, which he did not regard as obvious.

⁴⁷ D'Alembert had already discussed fluid resistance in part III of his treatise of 1744. There he used a molecular model in which momentum was transferred by impact from the moving body to a layer of hard molecules. He believed, however, that this molecular process would be negligible if the fluid molecules were too close to each other, for instance when fluid was forced through the narrow space between the body and a containing cylinder. In this case (1744: 205–206), he assumed a parallel-slice flow and computed the fluid pressure on the body through Bernoulli's law. For a head-tail symmetric body, this pressure does not contribute to the resistance if the flow has the same symmetry. After noting this difficulty, d'Alembert evoked the observed stagnancy of the fluid behind the body to retain only the Bernoulli pressure on the prow.

tains the equilibrium equation $\mathbf{f} = \nabla P$ that makes P the internal pressure! With d’Alembert’s own dynamical principle, one then reaches the equation of motion

$$\mathbf{f} - \rho \frac{d\mathbf{v}}{dt} = \nabla P, \quad (30)$$

which is nothing but Euler’s second equation. But d’Alembert did not proceed along these lines, and rather wrote equations of motion not involving internal pressure.⁴⁸

IV. EULER’S EQUATIONS

We finally turn to Euler himself for whom we shall be somewhat briefer than we have been with the Bernoullis and d’Alembert whose papers are not easily accessible to the untrained modern reader; not so with Euler. “Lisez Euler, lisez Euler, c’est notre maître à tous” (Read Euler, read Euler, he is the master of us all) used to say Pierre-Simon Laplace.⁴⁹

Pressure

After Euler’s arrival in Berlin he wrote a few articles on hydraulic problems, one of which was motivated by his participation in the design of the fountains of Frederick’s summer residence Sanssouci. In these works of 1750–51, Euler obtained the equation of motion for parallel-slice pipe flow by directly relating the acceleration of the fluid elements to the combined effect of the pressure gradient and gravity. He thus obtained the differential version

$$\frac{dv}{dt} = g - \frac{dP}{dz} \quad (31)$$

of Johann Bernoulli’s equation (7) for parallel-slice efflux. Therefrom he derived the generalization (8) of Bernoulli’s law to non-permanent flow, which he applied to evaluate the pressure surge in the pipes that would feed the fountains of Sanssouci.⁵⁰

Although d’Alembert had occasionally used this kind of reasoning in his theory of winds, it was new in a hydraulic context. As we saw, the Bernoullis did not rely on internal pressure in their own derivations of the equations of fluid motion. In contrast, Euler came to regard internal pressure as a key concept for a Newtonian approach to the dynamics of continuous media.

In a memoir of 1750 entitled “Découverte d’un nouveau principe de mécanique,” he claimed that the true basis of continuum mechanics was Newton’s second law applied to the infinitesimal elements of bodies. Among the forces acting on the elements he included “connection forces” acting on the boundary of the elements. In the case of fluids, these internal forces were to be identified to the pressure.⁵¹

Euler’s first attempt to apply this approach beyond the approximation of parallel-slices was a memoir on the motions of rivers written around 1750–1751. There he analyzed steady two-dimensional flow into filets and described the fluid motion through the Cartesian coordinates of a fluid particle expressed as functions of time and of a filet-labeling parameter (a partial anticipation of the so-called Lagrangian picture). He wrote partial differential equations expressing the incompressibility condition and his new principle of continuum dynamics. Through a clever combination of these equations, he obtained for the first time the Bernoulli law along the stream lines of an arbitrary steady incompressible flow. Yet he himself judged he had reached a dead end, for he could not solve any realistic problem of river flow in this manner.⁵²

The Latin memoir

An English translation of the latin memoir will be included in these Proceedings.

This relative failure did not discourage Euler. Equipped with his new principle of mechanics and probably stimulated by the two memoirs of d’Alembert, which he had reviewed, he set out to formulate the equations of fluid mechanics in full generality. A memoir entitled “De motu fluidorum in genere” was read in Berlin on 31 August 1752 and published under the title “Principia motus fluidorum” in St. Petersburg in 1761 as part of the 1756–1757 proceedings. Here Euler obtained the general equations of fluid motion for an incompressible fluid in terms of the internal pressure P and the Cartesian coordinates of the velocity \mathbf{v} .⁵³

In the first part of the paper he derived the incompressibility condition. For this he studied the deformation during a time dt of a small triangular element of water (in two dimension) and of a small triangular pyramid (in three dimensions). The method here is a slight generalization of what d’Alembert did in his memoir of 1749 on the resistance of fluids. Euler obtained, in his

⁴⁸ In this light, d’Alembert’s later neglect of Euler’s approach should not be regarded as a mere expression of rancor.

⁴⁹ Reported by Libri 1846: 51.

⁵⁰ Euler 1752. On the hydraulic writings, cf. Truesdell 1954: XLI–XLV; Ackeret 1957. On Euler’s work for the fountains of Sanssouci, cf. Eckert 2002, 2008. As Eckert explains, the failure of the fountains project and an ambiguous letter of the King of Prussia to Voltaire have led to the myth of Euler’s incapacity in concrete matters.

⁵¹ Euler 1750: 90 (the main purpose of this paper was the derivation of the equations of motion of a solid).

⁵² Euler 1760; Truesdell 1954: LVIII–LXII.

⁵³ Euler 1756–1757. Cf. Truesdell 1954: LXII–LXXV. D’Alembert’s role (also the Bernoullis’s and Clairaut’s) is acknowledged by Euler somewhat reluctantly in a sentence at the beginning of the third memoir cited in epigraph to the present paper.

own notation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \quad (32)$$

In the second part of the memoir, he applied Newton's second law to a cubic element of fluid subjected to the gravity \mathbf{g} and to the pressure P acting on the cube's faces. By a now familiar reasoning this procedure yields (for unit density) in modern notation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{g} - \nabla P. \quad (33)$$

Euler then eliminated the pressure gradient (basically by taking the curl) to obtain what we now call the vorticity equation:

$$\left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right] (\nabla \times \mathbf{v}) - [(\nabla \times \mathbf{v}) \cdot \nabla] \mathbf{v} = \mathbf{0}, \quad (34)$$

in modern notation. He then stated that "It is manifest that these equations are satisfied by the following three values $[\nabla \times \mathbf{v} = \mathbf{0}]$ in which is contained the condition provided by the consideration of the forces [i.e. the potential character of the r.h.s. of (33)]". He thus concluded that the velocity was potential, repeating here d'Alembert's mistake of confusing a necessary condition with a sufficient condition. This error allowed him to introduce what later fluid theorists called the velocity potential, that is, the function $\varphi(\mathbf{r})$ such that $\mathbf{v} = \nabla \varphi$. Equation (33) may then be rewritten as

$$\frac{\partial}{\partial t} (\nabla \varphi) + \frac{1}{2} \nabla (v^2) = \mathbf{g} - \nabla P. \quad (35)$$

Spatial integration of this equation yields a generalization of Bernoulli's law:

$$P = \mathbf{g} \cdot \mathbf{r} - \frac{1}{2} v^2 - \frac{\partial \varphi}{\partial t} + C, \quad (36)$$

wherein C is a constant (time-dependence can be absorbed in the velocity potential). Lastly, Euler applied this equation to the flow through a narrow tube of variable section to retrieve the results of the Bernoullis.

Although Euler's Latin memoir contained the basic hydrodynamic equations for an incompressible fluid, the form of exposition was still in flux. Euler frequently used specific letters (coefficients of differential forms) for partial differentials rather than Fontaine's notation, and he measured velocities and acceleration in gravity-dependent units. He proceeded gradually, from the simpler two-dimensional case to the fuller three-dimensional case. His derivation of the incompressibility equation was more intricate than we would now expect. And he erred in believing in the general existence of a velocity potential. These characteristics make Euler's Latin memoir a transition between d'Alembert's fluid dynamics and the fully modern foundation of this science found in the French memoirs.⁵⁴

⁵⁴ Cf. Truesdell 1954: LXII–LXXXV.

The French memoirs

An English translation of the second french memoir will be included in these Proceedings.

The first of these memoirs "Principes généraux de l'état d'équilibre des fluides" is devoted to the equilibrium of fluids, both incompressible and compressible. Euler realized that his new hydrodynamics contained a new hydrostatics based on the following principle: the action of the contiguous fluid on a given, internal element of fluid results from an isotropic, normal pressure P exerted on its surface. The equilibrium of an infinitesimal element subjected to this pressure and to the force density \mathbf{f} of external origin then requires

$$\mathbf{f} - \nabla P = 0. \quad (37)$$

As Euler showed, all known results of hydrostatics follow from this simple mathematical law.⁵⁵

The second French memoir, "Principes généraux du mouvement des fluides," is the most important one. Here Euler did not limit himself to the incompressible case and obtained the "Euler's equations" for compressible flow:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (38)$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{\rho} (\mathbf{f} - \nabla P), \quad (39)$$

to which a relation between pressure, density, and heat must be added for completeness.⁵⁶

The second French memoir is not only the coronation of many decades of struggle with the laws of fluid motion by the Bernoullis, d'Alembert and Euler himself, it contains much new material. Among other things, Euler now realized that $\nabla \times \mathbf{v}$ needed not vanish, as he had assumed in his Latin memoir, and gave an explicit example of incompressible vortex flow in which it did not.⁵⁷ In a third follow-up memoir entitled "Continuation des recherches sur la théorie du mouvement des fluides", he showed that even if it did not vanish Bernoulli's law remained valid along any stream line of a steady incompressible flow (as he had anticipated on his memoir of 1750–1751 on river flow). In modern terms: owing to the identity

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}), \quad (40)$$

the integration of the convective acceleration term along a line of flow eliminates $\nabla \times \mathbf{v}$ and contributes the $v^2/2$ term of Bernoulli's law.⁵⁸

⁵⁵ Euler 1755a.

⁵⁶ Euler 1755b: 284/63, 286/65. Cf. Truesdell 1954: LXXXV–C.

⁵⁷ As observed by Truesdell (1954: XC–XCI), in Section 66 Euler reverts to the assumption of non-vortical flow, a possible leftover of an earlier version of the paper.

⁵⁸ Euler 1755c: 345/117.

In his second memoir, Euler formulated the general problem of fluid motion as the determination of the velocity at any time for given values of the impressed forces, for a given relation between pressure and density, and for given initial values of fluid density and the fluid velocity. He outlined a general strategy for solving this problem, based on the requirement that the form $(\mathbf{f} - \rho \dot{\mathbf{v}}) \cdot d\mathbf{r}$ should be an exact differential (in order to be equal to the pressure differential). Then he confined himself to a few simple, soluble cases, for instance uniform flow (in the second memoir), or flow through a narrow tube (in the third memoir). In more general cases, he recognized the extreme difficulty of integrating his equations under given boundary conditions:⁵⁹

We see well enough . . . how far we still are from a complete knowledge of the motion of fluids, and that what I have explained here contains but a feeble beginning. However, all that the Theory of fluids holds, is contained in the two equations above [Eq. (1)], so that it is not the principles of Mechanics which we lack in the pursuit of these researches, but solely Analysis, which is not yet sufficiently cultivated for this purpose. Thus we see clearly what discoveries remain for us to make in this Science before we can arrive at a more perfect Theory of the motion of fluids.

V. CONCLUSIONS

In retrospect, Euler was right in judging that his “two equations” were the definitive basis of the hydrodynamics of perfect fluids. He reached them at the end of a long historical process of applying dynamical principles to fluid motion. An essential element of this evolution was the recurrent analogy between the efflux from a narrow vase and the fall of a compound pendulum. Any dynamical principle that solved the latter problem also solved the former. Daniel Bernoulli appealed to the conservation of live forces; Johann Bernoulli to Newton’s second law together with the idiosyncratic concept of *translatio*; d’Alembert to his own dynamical principle of the equilibrium of destroyed motions. With this more general principle and his feeling for partial differentials, d’Alembert leapt from parallel-slice flows to higher problems that involved two-dimensional anticipations of Euler’s equations. Although his method implicitly contained a general derivation of these equations in the incompressible case, his geometrical style and his abhorrence of internal forces prevented him from taking this step.

Despite d’Alembert’s reluctance, another important element of this history turns out to be the rise of the concept of internal pressure. So to say, the door on the way to general fluid mechanics opened with two different keys: d’Alembert’s principle, or the concept of internal pressure. D’Alembert (and Lagrange) used the first key, and introduced internal pressure only as a derivative concept.

Euler used the second key, and ignored d’Alembert’s principle. As Euler guessed (and as d’Alembert suggested *en passant*), Newton’s old second law applies to the volume elements of the fluid, if only the pressure of fluid on fluid is taken into account. Euler’s equations derive from this deceptively simple consideration, granted that the relevant calculus of partial differentials is known. Altogether, we see that hydrodynamics rose through the symbiotic evolution of analysis, dynamical principles, and physical concepts. Euler pruned unnecessary and unclear elements in the abundant writings of his predecessors, and combined the elements he judged most fundamental in the clearest and most general manner. He thus obtained an amazingly stable foundation for the science of fluid motion.

The discovery of sound foundations only marks the beginning of the life of a theory. Euler himself suspected that the integration of his equations would in general be a formidable task. It soon became clear that their application to problems of resistance or retardation led to paradoxes. In the following century, physicists struggled to solve these paradoxes by various means: viscous terms, discontinuity surfaces, instabilities. A quarter of a millennium later some very basic issues remain open, as many contributions to this conference amply demonstrate.

Acknowledgments

We are grateful to G. Grimberg, W. Pauls and two anonymous reviewers for many useful remarks. We also received considerable help from J. Bec and H. Frisch.

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⁵⁹ Euler 1755b: 315/91.

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