Mass transport in adhesive flows

Andreĭ Sobolevskiĭ

Physics Department
M.V. Lomonosov University of Moscow

Nonlinear Cosmology Workshop,
Nice 2006

Outline

1. Can shocks in solutions to the Burgers equation model collapsed structures adequately?

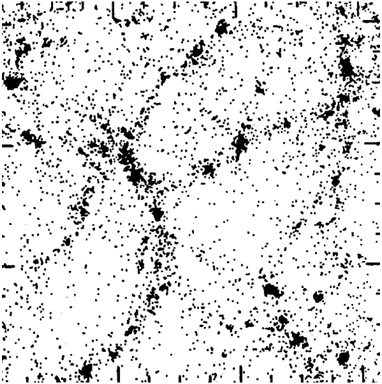
Outline

1. Can shocks in solutions to the Burgers equation model collapsed structures adequately?

2. Yes! (A variational construction of mass transport in shocks)

Collapsed structures

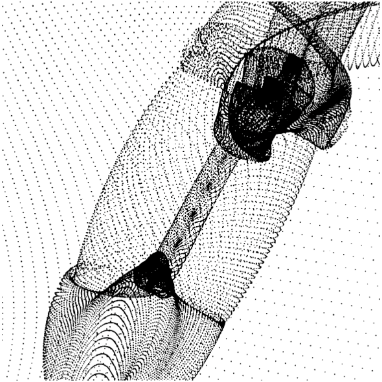
- Clusters
- Filaments
- Walls



(Weinberg, Gunn 1990)

Collapsed structures

- Clusters
- Filaments
- Walls



(Melott, Shandarin 1989)

The Burgers equation

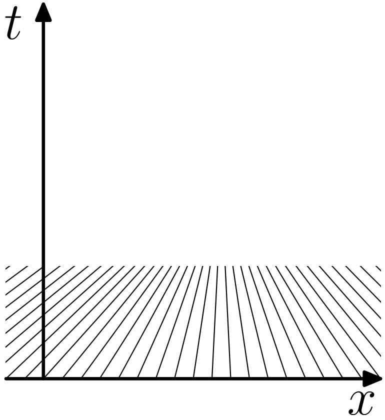
- Fluid particles move ballistically:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla) u = 0 \quad (\nabla \times u = 0)$$

- (Zel'dovich 1970, . . . many others)

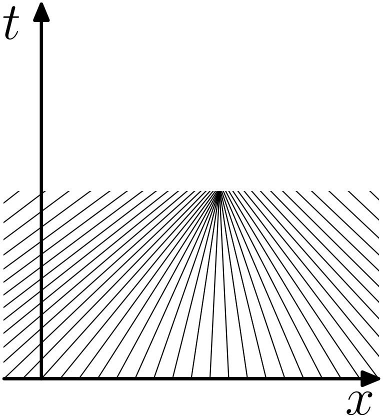
Crossing of trajectories

Before crossing: smooth solution



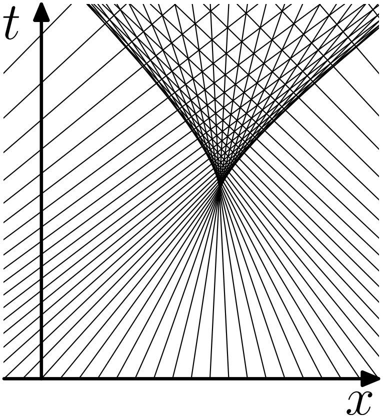
Crossing of trajectories

Trajectories cross. . .



Crossing of trajectories

After crossing:
multi-valued
solution?



The Burgers equation revisited

- Fluid particles move ballistically:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla)u = 0 \quad (\nabla \times u = 0)$$

The Burgers equation revisited

- Fluid particles move ballistically:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla) u = 0 \quad (\nabla \times u = 0)$$

- $u = \nabla \Phi$: $\quad \dfrac{\partial \Phi}{\partial t} + \dfrac{1}{2} |\nabla \Phi|^2 = 0$

The Hamilton–Jacobi equation

$$\frac{\partial \Phi}{\partial t} + H(\nabla \Phi) = 0$$

- $H(p) = \frac{1}{2}|p|^2$ is the Hamiltonian

The Hamilton–Jacobi equation

$$\frac{\partial \Phi}{\partial t} + H(\nabla \Phi) = 0$$

- $H(p) = \frac{1}{2}|p|^2$ is the Hamiltonian
- The Lagrangian:

$$L(v) = \max_p (v \cdot p - H(p))$$

The Hamilton–Jacobi equation

$$\frac{\partial \Phi}{\partial t} + H(\nabla \Phi) = 0$$

- $H(p) = \frac{1}{2}|p|^2$ is the Hamiltonian
- The Lagrangian:
$$L(v) = \max_p (v \cdot p - H(p))$$
- $L(v) + H(p) \geq v \cdot p$:
$$\text{equality if } v = \nabla_p H(p)$$

Φ is the action function

$$\frac{\partial \Phi}{\partial t} + H(\nabla \Phi) = 0$$

Directional derivative of Φ if $\dot{r} = v$:

$$\frac{d\Phi}{dt}[v] = v \cdot \nabla \Phi + \frac{\partial \Phi}{\partial t}$$

Φ is the action function

$$\frac{\partial \Phi}{\partial t} + H(\nabla \Phi) = 0$$

Directional derivative of Φ if $\dot{r} = v$:

$$\frac{d\Phi}{dt}[v] = v \cdot \nabla \Phi - H(\nabla \Phi)$$

Φ is the action function

$$\frac{\partial \Phi}{\partial t} + H(\nabla \Phi) = 0$$

Directional derivative of Φ if $\dot{r} = v$:

$$\frac{d\Phi}{dt}[v] = v \cdot \nabla \Phi - H(\nabla \Phi)$$

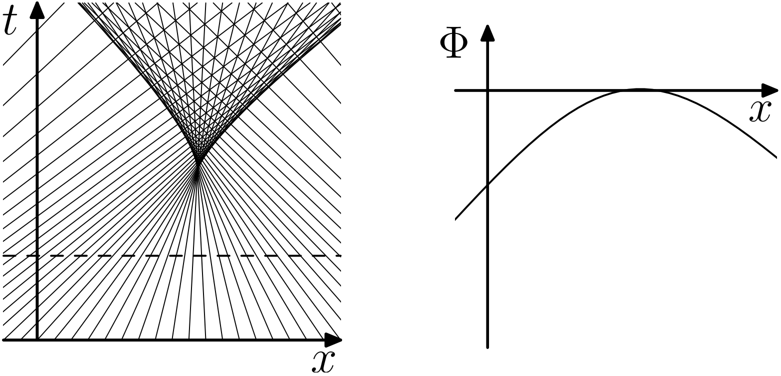
$$= L(\dot{r}) \quad \text{if } v = \dot{r} = \nabla_p H(\nabla \Phi)$$

The Min Action principle...

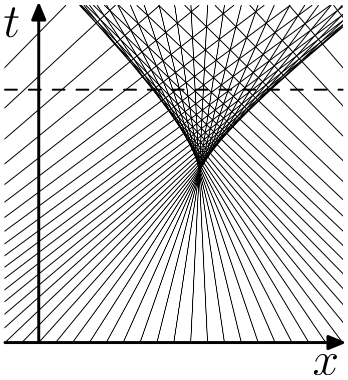
$$\frac{\partial \Phi}{\partial t} + H(\nabla \Phi) = 0$$

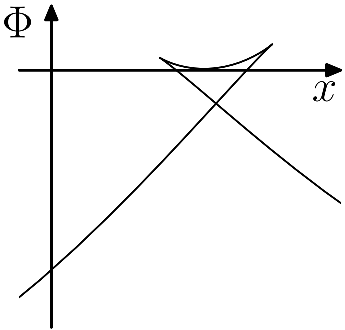
$$\Phi(r, t) = \min_{x(s):\, x(t)=r} \left(\Phi(x(s=0), 0) + \int_0^t L(\dot{x}(s))\, ds \right)$$

. . . and how it works



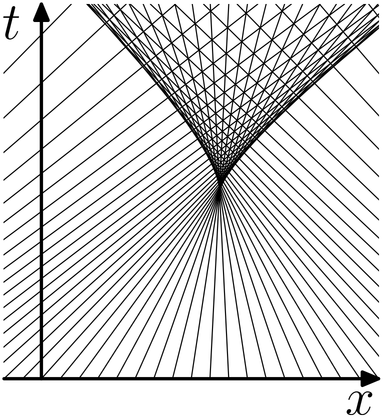
…and how it works





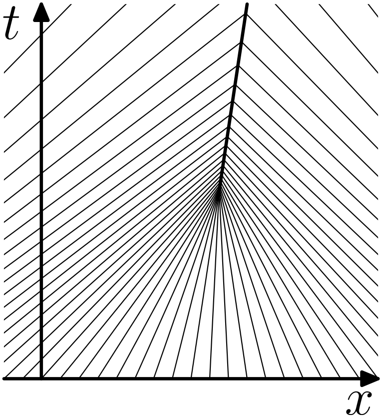
“Cutting” of trajectories

A “collapsed structure” appears:



“Cutting” of trajectories

A “collapsed structure” appears:



Trajectories extendable?

Question

Can one extend the trajectories of fluid particles beyond crossing the shock or do particles disappear on shocks?

Outline

1. Can shocks in solutions to the Burgers equation model collapsed structures adequately?

2. Yes! (A variational construction of mass transport in shocks)

The viscous limit

$$\frac{\partial \Phi^\mu}{\partial t} + \frac{1}{2} |\nabla \Phi^\mu|^2 = \mu \Delta \Phi^\mu$$

- As $\mu \to 0$, Φ^μ tends to the minimal action solution

The viscous limit

$$\frac{\partial \Phi^\mu}{\partial t} + \frac{1}{2} |\nabla \Phi^\mu|^2 = \mu \Delta \Phi^\mu$$

- As $\mu \to 0$, Φ^μ tends to the minimal action solution
- Where do the particle trajectories $\dot{r} = u^\mu(r, t) = \nabla \Phi^\mu(r, t)$ tend?

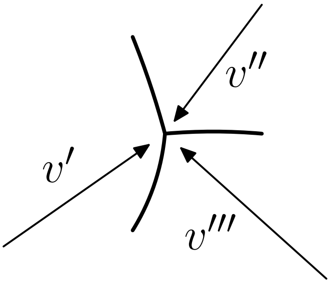
Motion inside shocks

Ilya Bogaevsky (math-ph/0407073)

The limiting trajectories of fluid particles as $\mu \to 0$ define a velocity field within collapsed structures. This velocity field can be constructed explicitly from $\Phi(r, t)$.

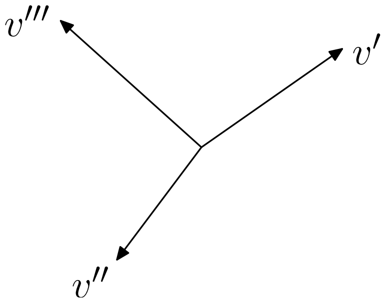
Motion inside shocks

Flow around a discontinuity in the r space:



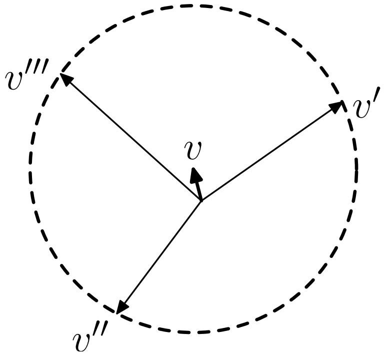
Motion inside shocks

Velocities in the v space:



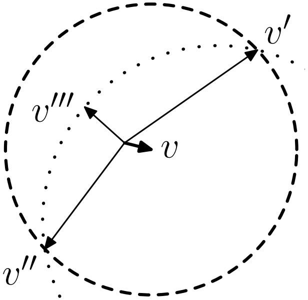
Motion inside shocks

Center of the smallest circle (sphere) containing v', v'', v''':



Motion inside shocks

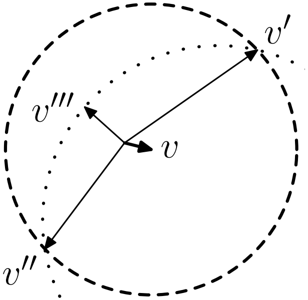
Center of the smallest circle (sphere) containing v', v'', v''':



Motion inside shocks

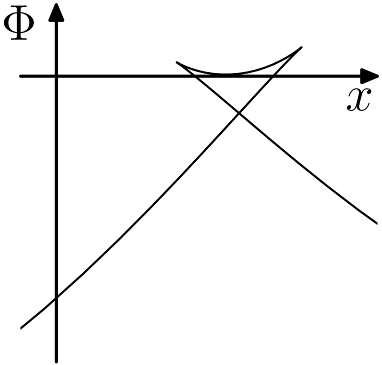
Question

Why the smallest circle?



Structure of Φ at shocks

At a shock, Φ is a minimum of several branches:



Structure of Φ at shocks

$$\Phi = \min(\Phi_1, \ldots, \Phi_m)$$

$$\frac{\partial \Phi_i}{\partial t} + H(\nabla \Phi_i) = 0$$

Non-minimizing trajectories

- A trajectory coming to a shock no longer minimizes the action

Non-minimizing trajectories

- A trajectory coming to a shock no longer minimizes the action
- *Idea:* try extending the trajectory so that its “surplus action” is min:

$$L(v) - \frac{d\Phi}{dt}[v] \to \min$$

Non-minimizing trajectories

- The velocity v is determined from

$$\max_i \left(L(v) - \frac{d\Phi_i}{dt}[v] \right) \to \min$$

Non-minimizing trajectories

- The velocity v is determined from

$$\max_i \left(L(v) - \frac{d\Phi_i}{dt}[v] \right) \to \min$$

- $d\Phi/dt[v] = v \cdot \nabla\Phi - H(\nabla\Phi)$

Non-minimizing trajectories

$$\max_i \Big(L(v) + H(\nabla\Phi_i) - v \cdot \nabla\Phi_i \Big) \to \min$$

Non-minimizing trajectories

$$\max_i \Big(L(v) + H(\nabla\Phi_i) - v \cdot \nabla\Phi_i \Big) \to \min$$

For $H(p) = |p|^2/2$, $L(v) = |v|^2/2$:

$$\max_i \frac{1}{2} |v - \nabla\Phi_i|^2 \to \min$$

Open questions

- Describe the limiting mass distribution within shocks
- Prove this result for general convex $H(p)$