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Wavelet analysis of the asteroidal resonant motion

T.A. Michtchenko and D. Nesvorný

Instituto Astronômico e Geofísico, Universidade de São Paulo, Caixa Postal 9638, CEP 01065, São Paulo, Brazil

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Abstract. The wavelet analysis method applied to study a resonant dynamical system gives sufficiently precise information about temporal evolution of its independent frequencies. Using the special basis functions set which automatically changes its size and position with respect to frequency and time, wavelet transform allows the study of both weakly and strongly chaotic behavior.

Key words: chaos – celestial mechanics – methods: analytical – asteroids

1. Introduction

It is known that, depending on initial conditions, the resonant asteroidal problem may have both regular and chaotic solutions. The Fourier transform (FT) methods can be applied with great success to describe the regular behavior of the dynamical system (Michtchenko & Ferraz-Mello 1995). By means of Fourier transform, signal x(t) is generally described as the sum of an infinite set of sine functions with infinite duration:

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{-2\pi i f t} df, \tag{1}$$

where $X(f)$ is the frequency spectrum of the signal. The basis of Fourier transform consists of harmonic functions that are completely localized in frequency but completely unlocalized in time. Therefore, the spectral composition of the regular motion, when the independent frequencies are constant in time, is uniquely defined by the Fourier transform.

In the case of chaotic motion, the independent frequencies of the dynamical system vary in time, and FT is, in general, of limited use since it does not describe the time dependence of the frequency components. In the last years, Laskar (1990,1993a,1993b) proposed to investigate the chaotic behavior of a dynamical system by analysing the temporal evolution of its fundamental frequencies. The temporal evolution of the independent frequencies of the conservative dynamical system implies a diffusion of the orbit in phase space. Thus, for the quantitative analysis of the chaotic diffusion, a dual description of the non-stationary signals in both time and frequency is needed.

Send offprint requests to: T.A. Michtchenko

Laskar developed the frequency analysis method and first successfully applied it to study the dynamics of planets in the Solar System, showing that the outer planets have a regular motion while the inner ones are chaotic. Recently, his method has been applied to investigate the slow diffusion in phase space of the 2:1 asteroidal resonance problem (Nesvorný & Ferraz-Mello 1995).

Laskar's frequency analysis method is very effective in the cases where the independent frequencies vary sufficiently slowly, so that their good measure can be obtained by FT. This method belongs to the class of methods called windowed Fourier transform methods (WFT), based on the adaption of the classical tools aimed at studying the stationary signals. This approach performs a time-dependent spectral analysis considering a non-stationary signal as a sequence of quasi-stationary segments for which Fourier transform methods are relevant. In this transform the signal is multiplied by a window function before performing FT, such as

$$X(f) = \frac{1}{2T} \int_{-T}^{+T} x(t)\, e^{i f t}\, g(t) dt, \tag{2}$$

where $g(t)$ is a window filter defined on a finite time span $[-T, T]$.

The limitations of WFT are derived from the fact that the window size must be chosen *a priori*. When independent frequencies of the dynamical system vary rapidly in a large scale (for example, in the case of the overlap of various secondary resonances), the sampling of the signals by a fixed-size window becomes inadequate and leads to loss of precision of the frequency measure. To overcome this difficulty, we suggest to apply the wavelet analysis method, which automatically adjusts both the time size and the frequency size of applied window.

2. Description of the wavelet analysis method

In this work, a continuous wavelet transform is used due to, mainly, our interest in the precise determination of the temporal frequency evolution. We briefly overview in this section some mathematical properties of the wavelet basis functions. For more detailed information we refer to the following works: 1) by Flandrin (1987) on the methods of the non-stationary signal processing; 2) by Grossman *et al.* (1987) on continuous

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wavelet transform; and 3) by Bendjoya & Slezak (1993) on its applications to some dynamical systems.

The continuous wavelet transform (WT) of a real signal x(t) is defined as:

$$C(a,b) = K(a)\int_{-\infty}^{+\infty} x(t)\, g^*\left(\frac{t-b}{a}\right) dt, \tag{3}$$

where $g(t)$ is an analysing wavelet function, $*$ indicates the complex conjugate and $K(a)$ is a normalization factor. The wavelet basis functions are two-parameter functions which are obtained from a single function $g(t)$ by translations acting on the time variable (parameter b) and dilatation acting on both the time and frequency variables (scaling parameter a). If $g^*(t)$ is a function vanishing outside some interval $[t_{\min},t_{\max}]$, the WT procedure isolates the time segment of the length $a(t_{\max}-t_{\min})$ centered at ab from time series $x(t)$. Note, that, for different values of the scaling parameter a, the basis wavelet functions have different sizes but identical shape to the analysing function $g(t)$.

The WT in Eq. (3) is considered as a correlation between signal $x(t)$ and dilated function $g^*(t/a)$. Using the time scaling and time shifting properties of Fourier transform, we can represent the WT equivalently in the frequency domain as

$$C(a,b) = a\,K(a)\int_{-\infty}^{+\infty} X(f)\,G^*(af)\,e^{2\pi i f b}\, df, \tag{4}$$

where $X(f)$ and $G(f)$ are the Fourier spectra of the signal and of the analysing wavelet function, respectively. Representation of $C(a,b)$ in this form allows us to understand the band-pass frequency feature of the wavelet transform. Indeed, if $G^*(f)$ is a function vanishing outside some interval $[f_{\min},f_{\max}]$, Eq. (4) shows that the wavelet transform procedure isolates the frequency band from time series $x(t)$, i.e. it acts as the band-pass filter and the width of the band-pass is equal to $(f_{\max}-f_{\min})/a$.

The conditions imposed on the analysing wavelet function $g(t)$ are:

1) locality, i.e. $g(t)$ should be a rapidly vanishing function both in time and frequency and 2) admissibility, which is required for purpose of reconstruction of signal from its wavelet transform:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|G(f)|^2}{|f|}df < \infty.$$

The choice of the function introduced by Morlet

$$g(t) = e^{-t^2/2}e^{i w_m t} + e^{-w_m^2/2} \tag{5}$$

as the analysing wavelet function, is adequate for the analysis of the time-dependent frequencies of the oscillatory signals. Here, the term in w_m^2 insures the admissibility and is negligible for $w_m > 5$. In this paper, we assume that $w_m = 2\pi$. The Gaussian function is mostly suitable as the analysing wavelet function due to its following properties: 1) it has the best possible simultaneous localization in time and in frequency; 2) it is closed under Fourier transform, pointwise multiplication and convolution; 3) the transition from one to more dimensions is immediate. The Gaussian is a rapidly vanishing function, but, in the strong sense, it is not a localized function because its tail extends to infinity. However, the good localization properties of any Gaussian may be "forced" by truncation at the points where it has a zero of sufficiently high order, i.e. at the points $b \pm \Delta t$, where $e^{-(t-b)^2/2a^2} = \epsilon$ (ϵ is small). Therefore, by using Gaussian as the analysing wavelet, we have

a) locality in time, which is determined by the semi-width of the gaussian function

$$\Delta t = a\sqrt{2|\ln\epsilon|}. \tag{6}$$

The interval $[b-\Delta t,\ b+\Delta t]$ is a time domain that influences the function $C(f,b)$ at a given point of the (b,f)-plane. Therefore, if the analysed frequency exists in some finite time interval, the ends of the width Δt of this interval, obtained by means of wavelet transform, are "polluted";

b) locality in frequency, which is given by

$$\Delta f = \frac{\sqrt{2|\ln\epsilon|}}{2\pi} f, \tag{7}$$

where $f = 1/a$. The interval $[f-\Delta f,\ f+\Delta f]$ hereafter called as the band-pass interval, is a range of Fourier components of $x(t)$ which are found at the given point of the (b,f)-plane.

In this work, the WT of a real signal x(t) is obtained as

$$C(f,b) = \sqrt{\frac{2}{\pi}}\frac{1}{a}\int_{b-\Delta t}^{b+\Delta t} x(t)\, e^{-(t-b)^2/2a^2} e^{-2\pi i f (t-b)}\, dt. \tag{8}$$

In practice, the integral (8) is calculated by conventional quadrature procedures on the grid of the b and f trial values. The range of the f variation is obtained from the FFT power spectrum of $x(t)$. The wavelet transform $C(f,b)$ is a complex function of two parameters: time b and frequency f. Here we use two alternative representations of the results of WT analysis: the first one consists in plotting the modulus (amplitude) of $C(f,b)$ for the fixed value of trial frequency; and the second one consists in plotting in the (b,f)-plane the values of frequency corresponding to the maximum of $|C(f,b)|$ obtained at a fixed point of time b.

There exists always an error which depends on the sampling grid and on the truncation of the wavelet function given by Eq. (5). The truncation error may be evaluated using Eq. (8). Indeed, at a given time b, we obtain

$$|C(f,b)| = A\,\Phi\left(\frac{\Delta t\, f}{\sqrt{2}}\right), \tag{9}$$

where A is the amplitude of the analysed f oscillation, $[b-\Delta t,\ b+\Delta t]$ is the lifetime interval of f and

$$\Phi(y) = \sqrt{\frac{2}{\pi}}\int_0^{+y} e^{-\frac{t^2}{2}} dt$$

is a probability integral, calculated by its series representation formula. The values of $\Phi(y)$ range in the interval between 0 and 1, the latter being the limiting value for $y \to \infty$. Thus, the truncation of $g(t)$ reduces the actual value of A by a factor of $\Phi(\frac{\Delta t\, f}{\sqrt{2}})$.

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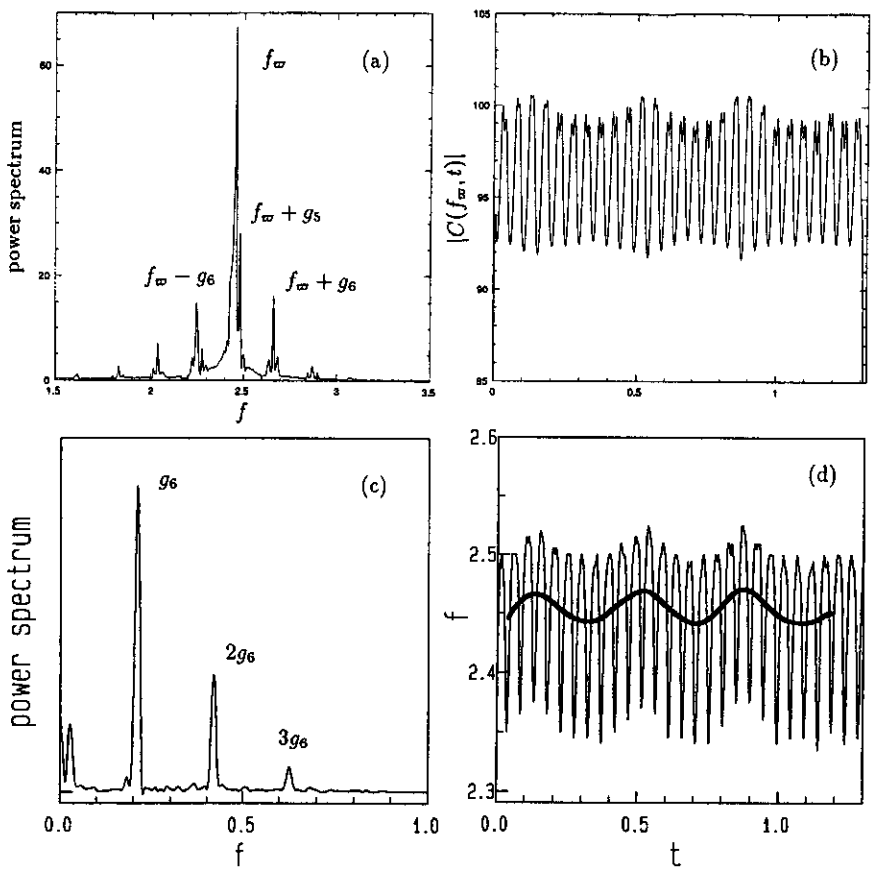

Fig. 1a–d. Orbit of regular motion: **a** FFT power spectrum of ϖ; **b** Modulus of $C(f_\varpi, t)$ vs time for fixed f_ϖ; **c** FFT power spectrum of $|C(f_\varpi, t)|$; **d** Frequency corresponding to max $|C(f, t)|$ at fixed t vs time. Time units are 10^6 years, frequency units are 10^{-4}/years

3. Applications of the wavelet analysis method

We have analysed by means of WT three orbits obtained by numerical integration of the asteroidal problem in the 2:1 resonance with Jupiter including the action of Saturn. The initial conditions of the fictitious asteroids were chosen in order to obtain the following cases: 1) one orbit of regular motion whose independent frequencies remain constant in the interval of integration; 2) one orbit of chaotic motion whose independent frequencies slowly vary in time and 3) one orbit of strongly chaotic motion. The frequency of the circulation of the longitude of perihelion f_ϖ was chosen as the WT analysis parameter.

3.1. Regular motion

In the case of a periodic signal of frequency f_ϖ, the WT coefficients may be obtained from (8) analytically and they are

$$C(f,b) = \frac{A}{i}\, e^{2\pi i f_\varpi b}\, e^{-\frac{(f-f_\varpi)^2}{2(\frac{f}{2\pi})^2}}.$$

$|C(f,b)|$ is time independent and is maximal at $f = f_\varpi$. Now we consider the case $\varpi(t) = A \sin 2\pi f_\varpi t + B \sin 2\pi(f_\varpi + \delta f)t$, where $(f_\varpi + \delta f)$ is a frequency from the band-pass interval; then, for a fixed f_ϖ, we have

$$|C(f_\varpi, b)|^2 = const + AB\, e^{-\frac{\delta f^2}{2(\frac{f_\varpi}{2\pi})^2}} \cos 2\pi\, \delta f\, b,$$

where the additive constant depends on A, B and δf. The modulus of the wavelet transform is a periodic function of frequency δf.

As an example which corresponds well to this case, one asteroidal orbit calculated in the planar model with initial conditions $a_0 = 3.2758088\ AU$, $e_0 = 0.2155$, $\Delta\varpi = 0°$ and $\sigma = 0°$ was analyzed. The Fourier power spectrum of ϖ over 1.3 Myr is shown in Fig. 1a. The access to the character of the motion is complicated by the presence of the spectral peaks, which surround the f_ϖ-line. Fig. 1b shows $|C(f_\varpi, b)|$ calculated numerically for the fixed f_ϖ, obtained from the Fourier spectrum. The regular character of motion becomes evident from the Fourier spectrum of $|C(f_\varpi, b)|$ (Fig. 1c); the spectral components of the lowest frequencies which are associated to g_5 and g_6 secular frequencies with periods equal to $\sim$ 300 000 and $\sim$ 45 000 years (Carpino *et al.* 1987), respectively, are seen in Fig. 1c. The time variation of the frequency corresponding to max $|C(f, b)|$, is traced in Fig. 1d by the thin curve. After filtering out the spectral components of frequency higher than 1/45 000 years, we obtained the slow f_ϖ variation with period $\sim$ 300 000 years, traced in Fig. 1d by the thick curve.

3.2. Chaotic motion

In order to illustrate the WT performance in the case of chaotic motion, it is convenient to distinguish between the weak and strong chaotic motion. But we must stress that the calculation procedures are similar in both cases. In this experiment, we have chosen the value of ϵ in Eq. (6) equal to $2\text{x}10^{-3}$; hence, the width of the analysing wavelet function in the time domain is $2\,\Delta t \approx 7\,a$. We assume that the frequency varies slowly, if it

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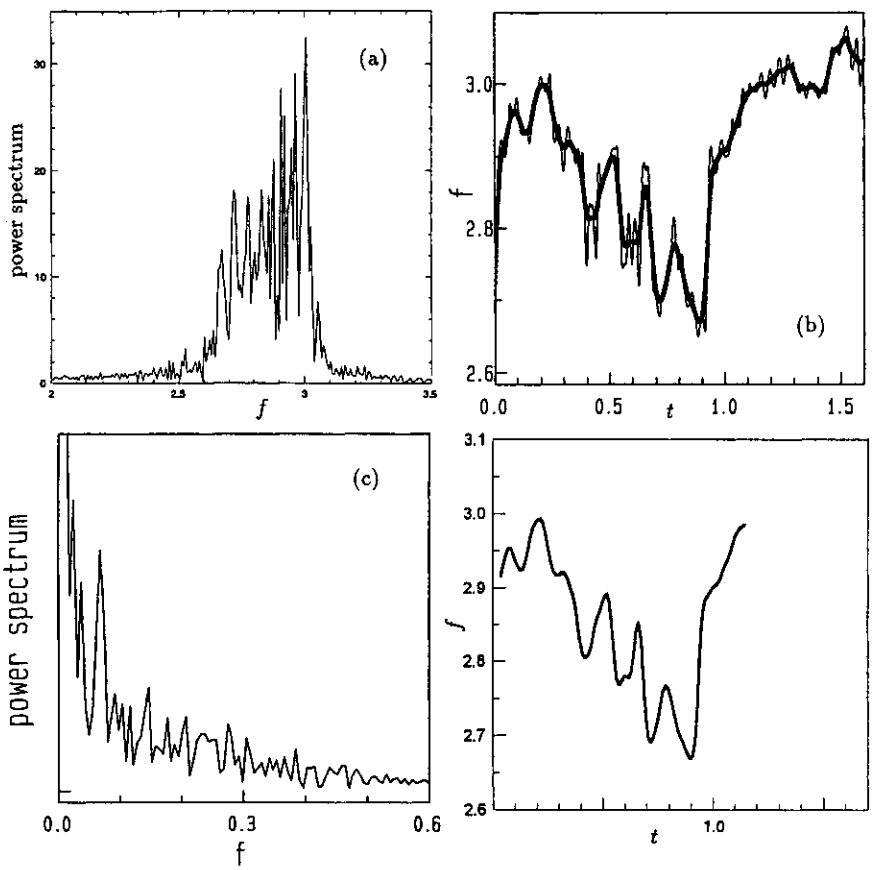

Fig. 2a–d. Orbit of weakly chaotic motion: **a** FFT power spectrum of ϖ; **b** Frequency corresponding to max $|C(f,t)|$ at fixed t vs time; **c** FFT power spectrum of max $|C(f,t)|$; **d** Same as for Fig. 2b, obtained by Laskar's (1990) frequency analysis. Time units are 10^6 years, frequency units are 10^{-4}/years

is approximately constant during at least seven oscillation periods; then, the asteroidal motion is considered weakly chaotic. In this case, the truncation error in the calculated amplitude of ϖ oscillation is equal to $5\text{x}10^{-4}A$. At a given time b, max $|C(f,b)|$ corresponds to the value of f_ϖ which may be obtained with a precision which is determined by the frequency grid chosen. In the case of strongly chaotic motion, when the frequency varies more rapidly, the calculated amplitude of the analyzed frequency is strongly reduced by the factor Φ; as a consequence, the amplitudes of the frequencies from the band-pass interval, given by (7), which are presented in the analysed time interval, become dominating. Hence, in the case of strongly chaotic motion, the WT procedure gives us the mean value of frequency over a time interval which corresponds to the time width of the analysing wavelet function.

1) One orbit of weakly chaotic motion was calculated in the planar model with initial conditions $a_0 = 3.2612404\,AU$, $e_0 = 0.2155$, $\sigma_0 = 0°$ and $\Delta\varpi_0 = 0°$. The Fourier power spectrum of ϖ over the time interval 1.3 Myr, shown in Fig. 2a, gives information about the interval of the frequency variation. The temporal evolution of the frequency corresponding to max $|C(f,b)|$ is shown in Fig. 2b by the thin curve. The Fourier spectrum of max $|C(f,b)|$ (Fig. 2c) confirms the chaotic character of motion. The smoothed thick curve in Fig. 2b obtained by filtering out the frequencies above 1/45 000 years, value approximately corresponding to a g_6 secular frequency, shows a slow diffusion of the f_ϖ frequency in time.

Fig. 2d represents the similar result obtained by application of the frequency analysis method developed by Laskar (1990). Similarity of Fig. 2b and Fig. 2d is apparent. In fact, from Eq. (2) and Eq. (8), it is noted that the frequency measures at a given time obtained by both methods differ only in the choice of applied window. Thus, in both cases, the frequency precision depends on the chosen frequency grid and on the quadrature procedure used. An important aspect is that the precision of the frequency measurement based on the Fourier transform methods is always limited by condition $\Delta t \Delta f = const$ obtained from Eq. (7) and Eq. (6).

2) To illustrate the case of strongly chaotic motion, we choose the initial conditions of the fictitious asteroid from the zone of the overlap of various secondary resonances ($a_0 = 3.2884\,AU$, $e_0 = 0.10$, $I_0 = 5°$, $\sigma_0 = 0°$ and $\Delta\varpi_0 = 0°$). The FFT power spectrum over 1.3 Myr (Fig. 3a) shows a large range of the f_ϖ variation. The various bands of spectral lines related to the presence of secondary resonances are indicated by the number of the corresponding resonance. The time evolution of the frequency corresponding to max $|C(f,b)|$ is plotted in Fig. 3b. The loci of the secondary resonances traced by horizontal lines are indicated by the corresponding ratio. The diffusion of the f_ϖ independent frequency to the regions of higher order secondary resonances is observed, as well as a temporary capture in the $\frac{5}{1}$ secondary resonance. In Fig. 3c we plotted max $|C(f,b)|$ as a function of time. According to Fig. 3c, the regions where f_ϖ varies rapidly, correspond to minimal values of max $|C(f,b)|$; in

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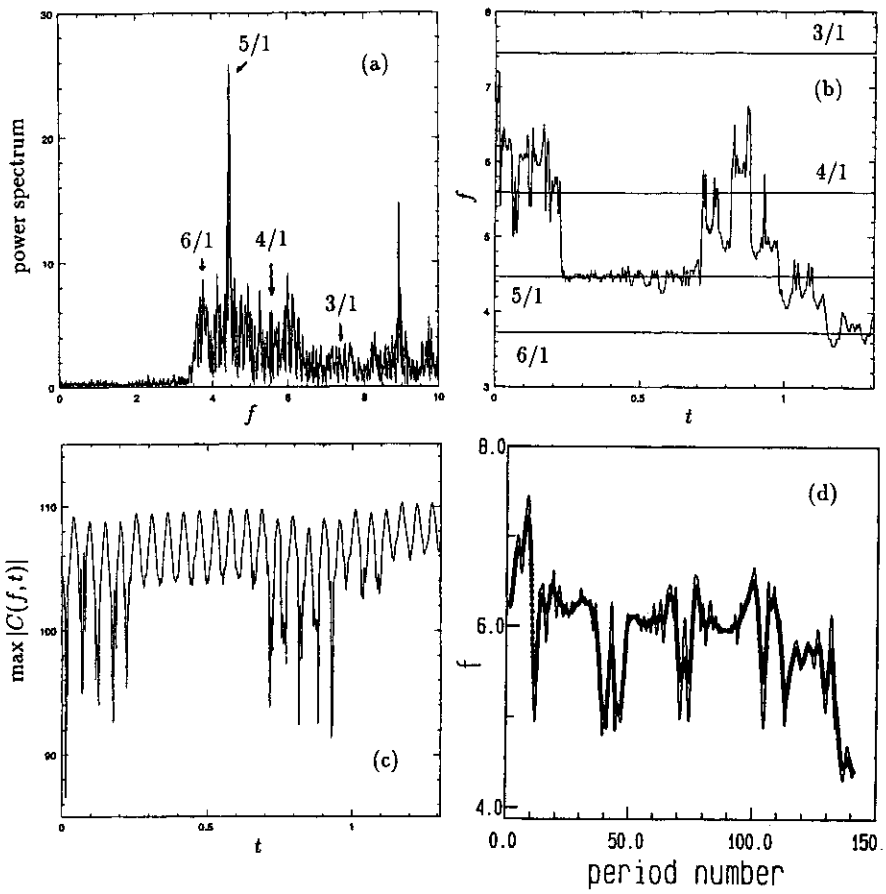

Fig. 3a–d. Orbit of strongly chaotic motion: **a** FFT power spectrum of ϖ; **b** Frequency corresponding to max $|C(f,t)|$ at fixed t vs time; **c** max $|C(f,t)|$ vs time; **d** Thin curve shows the periods of ϖ circulation measured approximately vs period number; thick curve is the same for Fig. 3b obtained by WT analysis over $\sim$ 220 000 years. Time units are 10^6 years, frequency units are 10^{-4}/years

contrast, in the region of capture in the $\frac{5}{1}$ secondary resonance, where f_{ϖ} varies slowly, max $|C(f,b)|$ is an approximately periodic function with a period corresponding to the frequency of the $\frac{5}{1}$ secondary resonant angle. We have also measured approximately the periods of the first 150 circulations of the angle ϖ ($\sim$ 220 000 years), which are plotted by the thin curve in Fig. 3d (abscissa is a circulation number). The thick curve represents the frequency variation obtained by the WT procedure and is shown in Fig. 5b in the corresponding time interval. It can be seen very clearly that the WT analysis has a smoothing property and gives the mean value of the frequency in the case of the strongly chaotic frequency variation.

4. Conclusion

The advantage of the WT against WFT is the efficiency of the non-stationary signal analysis caused by the special wavelet function set, which changes its size and position by dilatation and shifting. The wavelet transform decomposes the time series using a wavelet basis of functions that are localized both in time and frequency domains. The implementation of wavelet transform method is simple. The method shows a good performance when applied to either weakly or strongly chaotic motions and avoids the limitations of the methods based on the fixed-size window filter. Nevertheless, it has inevitable shortcomings due to the accuracy trade-offs between time and frequency, i.e. a good frequency resolution can only be achieved by means of a large window, which results in a poor time resolution and, on the contrary, a good time resolution implies short windows, which results in a poor frequency resolution.

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